Published by Institute of Physics Publishing for SISSA
Received: *December 9, 2005*
Accepted: *February 9, 2006*
Published: *February 27, 2006*

Noncommutative QCD, first-order-in-θ-deformed instantons and 't Hooft vertices

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Abstract: For commutative Euclidean time, we study the existence of field configurations that *a)* are formal power series expansions in $h\theta^{\mu\nu}$, *b)* go to ordinary (anti-)instantons as $h\theta^{\mu\nu} \to 0$, and *c)* render stationary the classical action of Euclidean noncommutative SU(3) Yang-Mills theory. We show that the noncommutative (anti-)self-dual equations have no solutions of this type at any order in $h\theta^{\mu\nu}$. However, we obtain all the deformations — called first-order-in-θ-deformed instantons — of the ordinary instanton that, at first order in $h\theta^{\mu\nu}$, satisfy the equations of motion of Euclidean noncommutative SU(3) Yang-Mills theory. We analyze the quantum effects that these field configurations give rise to in noncommutative SU(3) with one, two and three nearly massless flavours and compute the corresponding 't Hooft vertices, also, at first order in $h\theta^{\mu\nu}$. Other issues analyzed are the existence at higher orders in $h\theta^{\mu\nu}$ of topologically nontrivial solutions of the type mentioned above and the classification of the classical vacua of noncommutative SU(N) Yang-Mills theory that are power series in $h\theta^{\mu\nu}$.

Keywords: Nonperturbative Effects, Anomalies in Field and String Theories, Solitons Monopoles and Instantons.

Contents

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1. Introduction

Instantons play a major role in the understanding of the non-perturbative properties of QCD. The solution of the $\mathrm{U}(1)_A$ problem, the mass of the η' and the explanation of the spontaneous chiral symmetry breaking in QCD furnish instances of issues where instantons are the leading actors — see [1] and [2] and references therein. Two chief phenomena which are at the heart of instanton physics are following. First, instantons interpolate in Euclidean time between two classical vacuum states with winding numbers n and $n+1$, respectively, thus yielding the semi-classical contribution to the transition probability between these two classical vacuum states. Secondly, in the presence of massless quarks, the instanton transition leads to compulsory quark-anti-quark pair creation or, alternatively, turns a left handed quark into a right handed quark.

Instantons also occur in noncommutative U(N) Yang-Mills theories, in spite of the fact that, even classically, they are not invariant under scale transformations. It all started with the construction of instantons in noncommutative U(1) theory by the authors of ref. [3] — see also ref. [4]. These instantons have no counterpart in ordinary space. Then instantons in noncommutative U(2) theories were constructed [5 − 8], and thus was obtained the noncommutative counterpart of the celebrated BPST instanton [9]. (Multi)-Instantons in noncommutative U(N) gauge theories have also been constructed in refs. [10 − 15] and [16]. The physical effects of the noncommutative U(N) instantons have been analyzed in a number of papers. We shall just mention that the zero modes of the Dirac operator in a noncommutative instanton background have been studied in ref. [17] and that the quantum corrections around such types of backgrounds have been worked out for $N = 2$ supersymmetry in ref. [18].

Noncommutative QCD was constructed in ref. [19] as a part of the noncommutative standard model — see also refs. [20] and [23] and see refs. [21, 22] for other approaches. In the generalization of ordinary QCD of ref. [19], the noncommutative gauge field does not take values in the Lie algebra of SU(3), but rather in its enveloping algebra. Actually, the noncommutative fields are built from the ordinary fields with the help of the Seiberg-Witten map [24]. Thus it was circumvented what appears to be a shortcoming of the standard framework of noncommutative gauge theories, namely that it can only be applied to U(N) groups. Indeed, in this standard framework — see ref. [25] for a good introduction to the subject — the noncommutative gauge field unavoidably takes values in the Lie algebra of U(N), or direct products of such groups [26]. Some phenomenological [27, 28] and theoretical [29 − 31] properties of noncommutative gauge theories with SU(N) gauge groups have been investigated so far, but, a lot of work remains to be done. In particular, the study of the existence of instantons and, would they exist, the phenomena they give rise to, is, up to the best of our knowledge, a completely unexplored territory. This is in sharp contrast with the case of noncommutative U(N) theories. Note that in the case at hand, the SU(N) theory is not included in the U(N) case, since the noncommutative SU(N) gauge field does not take values in the Lie algebra of SU(N).

This paper is devoted — partially — to the study of the existence of field configurations in noncommutative SU(3) Yang-Mills theory that generalize the ordinary instanton field. We shall also analize the coupling between massless quarks of different chirality that these configurations give rise to and compute the corresponding 't Hooft vertices at first order in the noncommutative parameters $h\theta^{\mu\nu}$. $h\theta^{\mu\nu}$ define the noncommutative character of space, for the coordinates no longer commute but satisfy $[X^\mu, X^\nu] = i\, h\, \theta^{\mu\nu}$. h sets the noncommutative scale. Unless otherwise stated, we shall assume that Euclidean time is commutative — i.e., that $\theta^{4\,i} = 0,\ i = 1, 2$ and 3, in some reference system —, thus, upon Wick rotation the concept of evolution will be the ordinary one. Further, for this choice of $\theta^{\mu\nu}$, the Wick rotated action can be chosen to be at most quadratic in the first temporal derivative of the dynamical variables at any order in the expansion in $h\theta^{\mu\nu}$ and, thus, there is one conjugate momenta per ordinary field. This makes it possible to use simple

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Lagrangian and Hamiltonian methods to define the classical field theory and quantize it afterwards by using elementary and standard recipes. If time were not commutative the number of conjugate momenta will grow with the order of the expansion in h and then the Hamiltonian formalism will have to be generalized in some way or another [32, 33]. This generalization may affect the quantization process in some nontrivial way and deserves to be analyzed separately, perhaps along the lines laid out in ref. [32].

The layout of this article is as follows. In section 2, we look for — and conclude that there are none — solutions to the SU(N) noncommutative (anti)-self-duality equations that are formal power series in $h\theta^{\mu\nu}$, with $\theta^{4i}=0$, $i=1,2$ and 3. Section 3 deals with the construction of field configurations that go to the ordinary instanton as $h\theta^{\mu\nu}\to 0$ and that render stationary, at first order in $h\theta^{\mu\nu}$, the action of noncommutative SU(3) Yang-Mills theory. These field configurations will be called first-order-in-θ-deformed instantons. In section 4, we study the coupling between light left handed and right handed fermions that the field configurations found in the previous section produce and work out the appropriate 't Hooft vertices. We do this in this in theories with one, two and three light fermions. The two- and three-light fermions cases are relevant in connection with noncommutative QCD. In the last section, we summarize, draw conclusions and suggest how to carry on with the program started in this paper to include corrections at second order in $h\theta^{\mu\nu}$ or higher. The paper also includes five appendices. In appendix A, we consider an arbitrary $h\theta^{\mu\nu}$ and seek for solutions to the SU(N) noncommutative (anti)-self-duality equations that come as formal power series in $h\theta^{\mu\nu}$. The classical vacua of non-commutative SU(N) that are also formal power series in $h\theta^{\mu\nu}$ are found in appendix B, when time is commutative. Appendix C is devoted to the construction at first order in $h\theta^{\mu\nu}$ of the zero modes of the kinetic term of the quantum gauge field fluctuations in the background of a first-order-in-θ-deformed instanton. We also compute the zero mode of the $\theta-$deformed Dirac operator in that very background. In appendix D, we shall show that, when $\theta^{4i}=0$, $i=1,2,3$, no topologically nontrivial solutions can be found as power series in $h\theta^{\mu\nu}$ that solve the equations of motion of noncommutative SU(3) Yang-Mills theory. Several formulae used in the paper are collected in appendix E.

2. Noncommutative SU(N) instantons

A noncommutative SU(N) gauge field, $A_\mu[a_\nu]$, — see [20] — is a self-adjoint vector field that takes values in the enveloping algebra of the Lie algebra of SU(N) and that is obtained from a given ordinary SU(N) gauge field, a_μ, by means of a formal series expansion in powers of $h\theta^{\mu\nu}$ provided by the Seiberg-Witten map. As is well known, the Seiberg- Witten map is not unique [34−36], so that we shall call standard Seiberg-Witten map the straightforward generalization to SU(N) of the original expression introduced by Seiberg and Witten in ref. [37]. The standard form of the Seiberg-Witten map reads

$$A_\mu = a_\mu + \sum_{n=1}^{\infty}\frac{h^n}{n!}\,\frac{d^{n-1}}{dh^{n-1}}\left[\frac{dA_\mu}{dh}\right]\Bigg|_{h=0} = a_\mu - \frac{h}{4}\,\theta^{\alpha\beta}\{a_\alpha,\partial_\beta a_\mu + f_{\beta\mu}\} + O(h^2\theta^2), \qquad (2.1)$$

where

$$\frac{dA_\mu}{dh} = -\frac{1}{4}\,\theta^{\alpha\beta}\{A_\alpha, \partial_\beta A_\mu + F_{\beta\mu}\}_\star. \tag{2.2}$$

$f_{\mu\nu}$ stands for the ordinary field strength. The symbol $\star$ denotes the Moyal product of functions, i.e., $(f\star g)(x) = f(x)\exp(\frac{ih}{2}\theta^{\alpha\beta}\overleftarrow{\partial_\alpha}\overrightarrow{\partial_\beta})g(x)$, and $\{f,g\}_\star = (f\star g)(x) + (g\star f)(x)$.

From the previous A_μ, one constructs the noncommutative field strength, $F_{\mu\nu}[a_\sigma]$, as follows:

$$\begin{aligned} F_{\mu\nu}[a] &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star = f_{\mu\nu} + \sum_{n=1}^{\infty}\frac{h^n}{n!}\,\frac{d^{n-1}}{dh^{n-1}}\left[\frac{dF_{\mu\nu}}{dh}\right]\Bigg|_{h=0} \\ &= f_{\mu\nu} + \frac{h}{2}\theta^{\alpha\beta}\{f_{\mu\alpha}, f_{\nu\beta}\} - \frac{h}{4}\theta^{\alpha\beta}\{a_\alpha, (\partial_\beta + \mathfrak{D}_\beta)f_{\mu\nu}\} + O(h^2\theta^2). \end{aligned} \tag{2.3}$$

Here, $[A_\mu, A_\nu]_\star = A_\mu \star A_\nu - A_\nu \star A_\mu$ and

$$\frac{dF_{\mu\nu}}{dh} = \frac{1}{2}\theta^{\alpha\beta}\{F_{\mu\alpha}, F_{\nu\beta}\}_\star - \frac{1}{4}\theta^{\alpha\beta}\{A_\alpha, (\partial_\beta + \mathfrak{D}_\beta)F_{\mu\nu}\}_\star. \tag{2.4}$$

The action of a noncommutative SU(N) Yang-Mills theory is given by

$$\begin{aligned} S_{\rm NCYM} &= \frac{1}{2g^2}\int d^4x\ \mathrm{Tr}\,F_{\mu\nu}\star F_{\mu\nu} \\ &= \frac{1}{g^2}\int d^4x\ \mathrm{Tr}\left[\frac{1}{2}f_{\mu\nu}f_{\mu\nu} - \frac{h}{4}\theta_{\alpha\beta}f_{\alpha\beta}f_{\mu\nu}f_{\mu\nu} + h\theta_{\alpha\beta}\,f_{\mu\alpha}f_{\nu\beta}f_{\mu\nu}\right] + O(h^2\theta^2). \end{aligned} \tag{2.5}$$

We shall take a_μ to be in the fundamental representation of SU(N). We shall only consider ordinary gauge fields, a_μ, such that each term in the formal expansion on the r.h.s. of eq. (2.5) is finite. Thus, we shall impose the following boundary condition on a_μ:

$$a_\mu(x) \to ig(x)\partial_\mu g^\dagger(x) + O\Big(\frac{1}{|\,x\,|^2}\Big)\ \text{ as }\ |\,x\,| \to \infty. \tag{2.6}$$

$g(x)$ stands for an ordinary SU(N) gauge transformation.

It is then postulated that $S_{\rm NCYM}$ governs the dynamics of our SU(N) field theory on the four-dimensional noncommutative Euclidean space defined by $[\hat{X}^\mu, \hat{X}^\nu] = i\,h\,\theta^{\mu\nu}$, with $\theta^{i4} = 0$, $\forall i$.

Let us introduce the noncommutative dual field strength, $\tilde{F}_{\mu\nu}(x)$, and its ordinary counterpart:

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\,\epsilon_{\mu\nu\rho\sigma}\,F_{\rho\sigma}, \quad \tilde{f}_{\mu\nu} = \frac{1}{2}\,\epsilon_{\mu\nu\rho\sigma}\,f_{\rho\sigma}.$$

Then, a noncommutative SU(N) field $A_\mu[a_\sigma]$ has Pontrjagin index n if the following equation holds

$$n = \frac{1}{16\pi^2}\int d^4x\ \mathrm{Tr}\,(F_{\mu\nu}[a_\sigma]\star\tilde{F}_{\mu\nu}[a_\sigma]\,)(x). \tag{2.7}$$

It can be shown [31] that for ordinary gauge fields satisfying the boundary conditions in eq. (2.6), the Pontrjagin index of $A_\mu[a_\sigma]$ is equal to the Pontrjagin index of the ordinary field, a_σ, that defines the former as in eq. (2.1). Indeed,

$$\int d^4x\ \mathrm{Tr}\,(F_{\mu\nu}[a_\sigma]\star\tilde{F}_{\mu\nu}[a_\sigma]\,)(x) = \int d^4x\ \mathrm{Tr}\,f_{\mu\nu}(x)\tilde{f}_{\mu\nu}(x). \tag{2.8}$$

We shall say that $A_\mu[a_\sigma]$, defined by a given ordinary field a_μ as in eq. (2.1), is a noncommutative SU(N) instanton if it has Pontrjagin index — see eq. (2.7) — equal to one and it is a solution to the self-duality equation:

$$F_{\mu\nu}[a_\sigma] = \tilde{F}_{\mu\nu}[a_\sigma]. \tag{2.9}$$

It is not difficult to show that every noncommutative SU(N) instanton renders stationary the action in eq. (2.5). Indeed,

$$S_{\rm NCYM} = \mp\frac{1}{2g^2}\int d^4x\ {\rm Tr}\, F_{\mu\nu}\star\tilde{F}_{\mu\nu} + \frac{1}{4g^2}\int d^4x\ {\rm Tr}[(F_{\mu\nu}\pm\tilde{F}_{\mu\nu})\star(F_{\mu\nu}\pm\tilde{F}_{\mu\nu})]. \tag{2.10}$$

Both sides of the self-duality equation — eq. (2.9) — are defined as formal power series in $h\theta^{\mu\nu}$ — see eq. (2.3) — around the appropriate ordinary object: $f_{\mu\nu}$ or $\tilde{f}_{\mu\nu}$. Hence, one would like to find solutions to this equation that are formal series expansions in powers of $h\theta^{\mu\nu}$ around topologically nontrivial solutions to the ordinary self-duality equation $f_{\mu\nu} = \tilde{f}_{\mu\nu}$. We shall show below that no such solutions exist if $\theta^{i4} = 0$ in a given reference system.

Let a_μ, a solution to eq. (2.9), be given by the following formal power series in $h\theta^{\mu\nu}$:

$$a_\mu[h](x) = a_\mu^{(0)}(x) + \sum_{k=1}^{\infty} h^k\, a_\mu^{(k)}(x), \tag{2.11}$$

where $a_\mu^{(k)}(x)$ is a homogeneous polynomial in $\theta^{\mu\nu}$ of degree k whose coefficients are functions of x that take values in the Lie algebra of SU(N). Then, using the expression for $F_{\mu\nu}[a_\sigma]$ on the second line of eq. (2.3), one concludes that the following equations hold

$$\begin{aligned} f_{\mu\nu}^{(0)} &= \tilde{f}_{\mu\nu}^{(0)}, \\ \mathfrak{D}_\mu^{(0)} a_\nu^{(1)} - \mathfrak{D}_\nu^{(0)} a_\mu^{(1)} + \frac{1}{2}\theta^{\alpha\beta}\,\{f_{\mu\alpha}^{(0)}, f_{\nu\beta}^{(0)}\} &= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\left(\mathfrak{D}_\rho^{(0)} a_\sigma^{(1)} - \mathfrak{D}_\rho^{(0)} a_\sigma^{(1)} + \frac{1}{2}\,\theta^{\alpha\beta}\{f_{\rho\alpha}^{(0)}, f_{\sigma\beta}^{(0)}\}\right), \end{aligned} \tag{2.12}$$

where $\mathfrak{D}_\mu^{(0)} a_\nu^{(1)} = \partial_\mu a_\nu^{(1)} - i[a_\mu^{(0)}, a_\nu^{(1)}]$. Since $a_\nu^{(1)}(x)$ takes values in the Lie algebra of SU(N), not U(N), the trace over the SU(N) generators of both sides of the second equality in the previous equation yields

$$\theta^{\alpha\beta}\, f_{\mu\alpha}^{(0)\,a} f_{\nu\beta}^{(0)\,a} = \frac{1}{2}\,\theta^{\alpha\beta}\,\epsilon_{\mu\nu\rho\sigma} f_{\rho\alpha}^{(0)\,a} f_{\sigma\beta}^{(0)\,a}. \tag{2.13}$$

$f_{\mu\nu}^{(0)\,a}$ stand for the components of $f_{\mu\nu}^{(0)}$ in terms of the generators, T^a, of SU(N). Now, since $\theta^{i4} = 0$, we can always choose $\theta^{12} = \theta$ and $\theta^{21} = -\theta$ as only non-vanishing components of $\theta^{\mu\nu}$. For this $\theta^{\mu\nu}$ it is not difficult to show that the set of equations constituted by the first equality in eq. (2.12) and the identity in eq. (2.13) is equivalent to the following set of equations:

$$\begin{aligned} &f_{12}^{(0)\,a} = f_{34}^{(0)\,a}, \quad f_{13}^{(0)\,a} = -f_{24}^{(0)\,a}, \quad f_{14}^{(0)\,a} = f_{23}^{(0)\,a}, \\ &\sum_a (f_{12}^{(0)\,a})^2 + (f_{13}^{(0)\,a})^2 + (f_{14}^{(0)\,a})^2 = 0. \end{aligned} \tag{2.14}$$

From this set of equations one readily concludes that $a_\mu^{(0)}$ has vanishing field strength, so that it is gauge equivalent to the vanishing gauge field.

We have shown so far that when the standard Seiberg-Witten map — defined in eqs. (2.1) and (2.2) — is employed to define the noncommutative field strength $F_{\mu\nu}$, the self-duality equation — eq. (2.9) — has no solution of the type displayed in eq. (2.11). We shall show next that this state of affairs remains unaltered for the most general type of Seiberg-Witten map. At first order in $h\theta^{\mu\nu}$, the most general expression for the Seiberg-Witten map reads

$$\begin{aligned} A_\mu &= a_\mu - \frac{h}{4}\theta^{\alpha\beta}\{a_\alpha, \partial_\beta a_\mu + f_{\beta\mu}\} + \kappa_1\, h\theta^{\alpha\beta}\,\mathfrak{D}_\mu f_{\alpha\beta} + \kappa_2\, h\theta^{\alpha\beta}\,\mathfrak{D}_\mu[a_\alpha, a_\beta] \\ &\quad +\kappa_3\, h\theta_\mu{}^\beta \mathfrak{D}^\nu f_{\nu\beta} + O(h^2\theta^2), \end{aligned} \tag{2.15}$$

where κ_i, $i = 1, 2, 3$ are arbitrary real numbers. The noncommutative field strength for the previous noncommutative gauge field is given by

$$\begin{aligned} F_{\mu\nu} &= f_{\mu\nu} + \frac{h}{2}\theta^{\alpha\beta}\{f_{\mu\alpha}, f_{\nu\beta}\} - \frac{h}{4}\theta^{\alpha\beta}\{a_\alpha, (\partial_\beta + \mathfrak{D}_\beta)f_{\mu\nu}\} - i\kappa_1\, h\theta^{\alpha\beta}[f_{\mu\nu}, f_{\alpha\beta}] \\ &\quad -i\kappa_2\, h\theta^{\alpha\beta}[f_{\mu\nu}, [a_\alpha, a_\beta]] - \kappa_3\, h\big(\theta_\mu{}^\beta \mathfrak{D}_\nu \mathfrak{D}^\rho f_{\rho\beta} - \theta_\nu{}^\beta \mathfrak{D}_\mu \mathfrak{D}^\rho f_{\rho\beta}\big) + O(h^2\theta^2). \end{aligned}$$

Substituting this expression in both sides of eq. (2.9), one concludes that eq. (2.13) is not modified by the new terms in the previous $F_{\mu\nu}$. Hence, no solutions to the noncommutative self-duality equation can be found by using formal powers series in $h\theta^{\mu\nu}$ around ordinary fields with non-vanishing instanton number.

It is clear that the result we have obtained for the self-duality equation also carries over to the anti-self-duality equation:

$$F_{\mu\nu} = -\tilde{F}_{\mu\nu}.$$

The reader may wonder what would the situation be had we assumed an arbitrary $\theta^{\mu\nu}$. In appendix A we show that the self-duality equation defined by the standard Seiberg-Witten map has topologically nontrivial solutions of the type in eq. (2.11) if, and only if, $\theta^{\mu\nu}$ is self-dual: $\theta_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\theta^{\rho\sigma}$. Actually, these solutions are the ordinary gauge field configurations that are self-dual. Analogously, the anti-self-duality equation has solutions of the type in eq. (2.11) with non-vanishing Pontrjagin number if, and only if, $\theta^{\mu\nu}$ is anti-self-dual: $\theta_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\theta^{\rho\sigma}$. These solutions are ordinary field configurations that are anti-self-dual. In other words, if $\theta^{\mu\nu}$ is self-dual, the standard Seiberg-Witten map in eq. (2.1) maps ordinary (multi-)instantons into noncommutative (multi-)instantons and if $\theta^{\mu\nu}$ is anti-selfdual, the standard Seiberg-Witten map maps ordinary (multi-)anti-instantons into noncommutative (multi-)anti-instantons.

3. First-order-in-θ-deformed instantons

In this section we shall look for ordinary field configurations, $a_\mu[h](x)$, that render stationary the action of Euclidean noncommutative SU(3) Yang-Mills theory up to first order in $h\theta^{\mu\nu}$ and that admit a formal series expansion in powers of $h\theta^{\mu\nu}$. We shall further assume that these field configurations have a smooth dependence on the space coordinates, that they satisfy the boundary conditions in eq. (2.6) and that they go to the ordinary instanton as $h \to 0$.

From eq. (2.5), one derives the following equation of motion up to first order in $h\theta^{\mu\nu}$:

$$\mathrm{Tr}\, T^a \left[\mathfrak{D}_\mu f_{\mu\nu} + h\theta^{\alpha\beta} \left\{ f_{\mu\beta}, -\mathfrak{D}_\alpha f_{\nu\mu} + \frac{1}{2}\mathfrak{D}_\nu f_{\alpha\mu} \right\} \right] = 0 + O(h^2\theta^2). \tag{3.1}$$

Now, any ordinary SU(3) instanton, a_μ^{oinst}, has the form

$$a_\mu^{\mathrm{oinst}}(x) = U a_\mu^{\mathrm{BPST}}(x) U^\dagger,$$

where U is an arbitrary rigid SU(3) transformation and $a_\mu^{\mathrm{BPST}}(x)$ denotes the upper-left-hand corner embedding in SU(3) of the ordinary SU(2) instanton, which can be written as

$$a_\mu^{\mathrm{BPST}}(x) = \eta_{a\mu\nu} \frac{(x-x_0)_\nu}{(x-x_0)^2+\rho^2}\, T^a \tag{3.2}$$

in the regular gauge. T^a, $a = 1,2,3$ denote the upper-left-hand corner embedding of the SU(2) generators into the SU(3) generators and $\eta_{a\mu\nu}$ stands for the self-dual 't Hooft symbols [38].

Since eq. (3.1) is invariant under SU(3) transformations, we shall find first a solution to it of the form

$$a_\mu[h] = a_\mu^{\mathrm{BPST}} + h b_\mu + O(h^2\theta^2). \tag{3.3}$$

Then, we shall apply to this solution an arbitrary rigid SU(3) transformation. This type of solutions will be called first-order-in-θ-deformed instanton. b^μ in eq. (3.3) is an SU(3) Lie-algebra-valued smooth vector field which is linear in $\theta^{\mu\nu}$ and vanishes rapidly enough at infinity.

Let us substitute the previous $a_\mu[h]$ in eq. (3.1) and discard any contribution of order $h^2\theta^2$. Of course, the order h^0 contribution thus obtained is satisfied by construction: $\mathfrak{D}_\mu^{\mathrm{BPST}} f_{\mu\nu}^{\mathrm{BPST}} = 0$. The order $h\theta^{\mu\nu}$ contribution yields a non-homogeneous equation for b_μ:

$$\begin{aligned} &-i\,\mathrm{Tr}\, T^a[b_\mu, f_{\mu\nu}^{\mathrm{BPST}}] + \mathrm{Tr}\, T^a[\mathfrak{D}_\mu^{\mathrm{BPST}}\mathfrak{D}_\mu^{\mathrm{BPST}} b_\nu - \mathfrak{D}_\mu^{\mathrm{BPST}}\mathfrak{D}_\nu^{\mathrm{BPST}} b_\mu] \\ &\quad = \frac{1}{2}\ \mathrm{Tr}\, T^a\theta^{\alpha\beta}\{f_{\mu\beta}^{\mathrm{BPST}}, \mathfrak{D}_\alpha^{\mathrm{BPST}} f_{\nu\mu}^{\mathrm{BPST}} + \mathfrak{D}_\mu^{\mathrm{BPST}} f_{\nu\alpha}^{\mathrm{BPST}}\}. \end{aligned} \tag{3.4}$$

The action of $[T^a,]$, $a \in \{1,2,3\}$ over the generators of SU(3) defines four irreducible representations of SU(2): a spin one representation acting on the linear span of $\{T^1, T^2, T^3\}$, two spin 1/2 representations acting on the linear span of $\{T^4, T^5, T^6, T^7\}$ and a singlet acting on T^8. On the other hand, if $a, b \in \{1,2,3\}$, we have $\{T^a, T^b\} = \frac{1}{3}\delta^{ab}\mathrm{I\!I} + d_{abc}T^c|_{a,b\in\{1,2,3\}} = \frac{1}{3}\delta^{ab}\mathrm{I\!I} + \frac{1}{\sqrt{3}}\delta^{ab}T^8$. This last identity implies that the r.h.s. of eq. (3.4) is non-zero only for $a = 8$. Let us express b_ν as

$$b_\nu = b_\nu^{(1\ldots7)} + b_\nu^8 T^8, \quad b_\nu^{(1\ldots7)} = \sum_{c=1}^{7} b_\nu^c T^c. \tag{3.5}$$

Then, taking account that the action of $[T^a,]$, $a \in \{1,2,3\}$, does not mix the irreducible representations mentioned above, one concludes that the equation of motion for $b_\nu^{(1\ldots7)}$ decouples from that of b_ν^8. For $b_\nu^{(1\ldots7)}$ the equation of motion reads

$$(\mathfrak{D}_\mu^{\rm BPST}(\mathfrak{D}_\mu^{\rm BPST} b_\nu^{(1\ldots7)} - \mathfrak{D}_\nu^{\rm BPST} b_\mu^{(1\ldots7)}) - i[b_\mu^{(1\ldots7)}, f_{\mu\nu}^{\rm BPST}])^a = 0, \qquad a \in 1,\ldots,7, \tag{3.6}$$

whereas for b_ν^8, we have the following non-homogeneous equation:

$$(\Box\,\delta_{\mu\nu} - \partial_\mu\partial_\nu)\, b_\nu^8 = \sum_a \frac{\theta^{\alpha\beta}}{2\sqrt{3}} [f_{\mu\beta}^{{\rm BPST}\,a} (\mathfrak{D}_\alpha^{\rm BPST} f_{\nu\mu}^{\rm BPST} + \mathfrak{D}_\mu^{\rm BPST} f_{\nu\alpha}^{\rm BPST})^a]. \tag{3.7}$$

Let us first solve this second equation. The most most general solution to eq. (3.7) is of the form $b_\mu^8 = b_\mu^{8\,(hom)} + b_\mu^{8\,(part)}$, $b_\mu^{8\,(part)}$ being a particular solution to it and $b_\mu^{8\,(hom)}$ denoting the most general solution to the corresponding homogeneous equation

$$(\Box\,\delta_{\mu\nu} - \partial_\mu\partial_\nu)\, b_\nu^{8\,(hom)} = 0.$$

Any solution to the latter equation which is smooth and vanishes at infinity reads $b_\mu^{8\,(hom)} = \partial_\mu\phi$, where ϕ is an appropriate function. Recalling that T^8 commutes with T^a, $a \in \{1,2,3\}$, one concludes that this $b_\mu^{8\,(hom)}$ can always be generated by applying to $a_\mu[h]$ in eq. (3.3) the following gauge transformation: $g(x) = e^{ih\phi(x)T^8}$. We are thus left with the problem of finding a particular solution, $b_\mu^{8\,(part)}$, to eq. (3.7).

Let us assume that $b_\mu^{8\,(part)}$ satisfies the transversality condition $\partial_\mu b_\mu^{8\,(part)} = 0$ and has the following form:

$$b_\mu^{8\,(part)} = \theta_{\mu\nu}(x-x_0)_\nu f[(x-x_0)^2] - \tilde\theta_{\mu\nu}(x-x_0)_\nu g[(x-x_0)^2], \quad \tilde\theta_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\theta_{\rho\sigma}, \tag{3.8}$$

where x_0 is the centre of the ordinary BPST instanton. Then, eq. (3.7) boils down to the following equation to be satisfied by $f[(x-x_0)^2]$ and $g[(x-x_0)^2]$:

$$3y' + (x-x_0)^2 y'' = -\frac{48\rho^4}{\sqrt{3}[(x-x_0)^2+\rho^2]^5}, \qquad y = f,\, g.$$

A solution to this equation that is smooth and vanishes at infinity is given by

$$y(x) = f[(x-x_0)^2] = g[(x-x_0)^2] = \frac{2}{\sqrt{3}} \frac{r^2+3\rho^2}{(r^2+\rho^2)^3},$$

where $r = (x-x_0)^2$. Substituting the previous result in eq. (3.8), one finally gets that, modulo gauge transformations, the solution, b_μ^8, to eq. (3.7) that is smooth and vanishes at infinity reads

$$b_\mu^8 = b_\mu^{8\,(part)} = \frac{2}{\sqrt{3}}\left(\theta_{\mu\nu}(x-x_0)_\nu - \tilde\theta_{\mu\nu}(x-x_0)_\nu\right) \frac{r^2+3\rho^2}{(r^2+\rho^2)^3}. \tag{3.9}$$

Again, $r^2 = (x-x_0)^2$.

Let us now solve eq. (3.6). Using the fact that $f_{\mu\nu}^{\text{BPST}} = \tilde{f}_{\mu\nu}^{\text{BPST}}$ and the relation $[f_{\mu\nu},] = i[\mathfrak{D}_\mu, \mathfrak{D}_\nu]$, this equation can be turned into the following equality:

$$\left(\mathfrak{D}_\mu^{\text{BPST}}\left\{[\mathfrak{D}_\mu^{\text{BPST}} b_\nu^{(1\ldots7)} - \mathfrak{D}_\nu^{\text{BPST}} b_\mu^{(1\ldots7)}] - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}[\mathfrak{D}_\rho^{\text{BPST}} b_\sigma^{(1\ldots7)} - \mathfrak{D}_\sigma^{\text{BPST}} b_\rho^{(1\ldots7)}]\right\}\right)^a = 0,$$
$$a \in 1, \ldots, 7. \qquad (3.10)$$

We shall show next that any smooth $b_\mu^{(1\ldots7)}$ that vanishes at infinity is a solution to the previous equation if, and only if, it solves the following equality:

$$[\mathfrak{D}_\mu^{\text{BPST}} b_\nu^{(1\ldots7)} - \mathfrak{D}_\nu^{\text{BPST}} b_\mu^{(1\ldots7)}] - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}[\mathfrak{D}_\rho^{\text{BPST}} b_\sigma^{(1\ldots7)} - \mathfrak{D}_\sigma^{\text{BPST}} b_\rho^{(1\ldots7)}] = 0. \qquad (3.11)$$

Let $\Omega_{\mu\nu}$ denote the left hand side of the previous equation. Then eq. (3.10) reads

$$\mathfrak{D}_\mu^{\text{BPST}} \Omega_{\mu\nu} = 0. \qquad (3.12)$$

Since $\Omega_{\mu\nu}$ is an anti-symmetric and anti-self-dual object, it has the following representation in terms of the symmetric spinorial object $\Omega^{\alpha\beta}$: $\Omega_{\mu\nu} = i(\sigma_{\mu\nu})_{\alpha\beta}\,\Omega^{\alpha\beta}$, where $\sigma_{\mu\nu} = -\frac{1}{4i}(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu)$ with $\sigma_\mu = (\vec{\sigma}, i)$ and $\bar{\sigma}_\mu = (-\vec{\sigma}, i)$. In terms of $\Omega^{\alpha\beta}$, eq. (3.12) reads

$$\text{tr}\left(\sigma_\nu \mathfrak{D}_\mu^{\text{BPST}} \bar{\sigma}_\mu \Omega^\top\right) = 0,$$

where tr stands for the trace over the spinor indices and $\Omega^\top$ is the transpose of Ω with regard to the latter indices. Since $\{\sigma_\nu\}$ is an orthogonal basis of the 2×2 matrices, the previous equation is equivalent to

$$\mathfrak{D}_\mu^{\text{BPST}} \bar{\sigma}_\mu \Omega^\top = 0.$$

Applying $\mathfrak{D}_\nu^{\text{BPST}}\sigma_\nu$ to this equation, one concludes that

$$(\mathfrak{D}^{\text{BPST}})^2\,\Omega^\top = 0.$$

Indeed, just take into account that $\sigma_\mu\bar{\sigma}_\nu = -\delta_{\mu\nu} - 2i\sigma_{\mu\nu}$ and that $\sigma_{\mu\nu} f_{\mu\nu}^{\text{BPST}} = 0$. The latter equality is a consequence of $\sigma_{\mu\nu}$ being anti-self-dual and $f_{\mu\nu}^{\text{BPST}}$ being self-dual. Now, $(\mathfrak{D}^{\text{BPST}})^2$ is a positive definite operator, so that it has no normalizable non-vanishing eigenvectors with zero eigenvalue — see [39]. Hence,

$$\Omega^{\alpha\beta} = 0.$$

Recalling that $\Omega_{\mu\nu} = i(\sigma_{\mu\nu})_{\alpha\beta}\,\Omega^{\alpha\beta}$, we conclude that $\Omega_{\mu\nu}(x)$ — smooth anti-symmetric anti-self-dual object that vanishes at infinity — satisfies eq. (3.12) if, only if, $\Omega_{\mu\nu}(x) = 0$. We have thus shown that solving eq. (3.10) is equivalent to solving eq. (3.11) for smooth functions that vanish at infinity rapidly enough. Now, eq. (3.11) is the equation for the zero modes of the ordinary SU(3) instanton [40]. Hence, the $b_\nu^{(1\ldots7)}$ we are looking for are linear combinations of those zero modes with coefficients that depend linearly on $\theta^{\mu\nu}$, and can thus be obtained by deforming infinitesimally the collective coordinates of a given SU(3) ordinary instanton. Since this deformation yields another SU(3) instanton, we shall set, without loss of generality, $b_\nu^{(1\ldots7)}$ to zero. Substituting both this result and eq. (3.9) in eq. (3.5), and the result so obtained, in turn, in eq. (3.3), one gets the most general first-order-in-θ-deformed instanton, $a_\mu[h]^{(\mathrm{gen})}$, in the regular gauge:

$$a_\mu[h]^{(\mathrm{gen})}(x) = U a_\mu[h](x) U^\dagger. \tag{3.13}$$

Here, U is an arbitrary rigid SU(3) transformation that does not leave $a_\mu[h]$ invariant and $a_\mu[h]$ is given by

$$a_\mu[h] = a_\mu^{\mathrm{BPST}} + h\,T^8 \frac{2}{\sqrt{3}}\,[\theta_{\mu\nu} - \tilde{\theta}_{\mu\nu}](x-x_0)_\nu\,\frac{r^2+3\rho^2}{(r^2+\rho^2)^3}, \tag{3.14}$$

where a_μ^{BPST} is defined in eq. (3.2) and $r^2 = (x-x_0)^2$.

It can be shown that $a_\mu[h]$ has instanton number equal to 1 and that its contribution to the noncommutative Yang-Mills action in eq. (2.10) reads

$$S_{\mathrm{NCYM}} = \frac{8\pi^2}{g^2} + O(h^2\theta^2).$$

Hence, at the order we are working — first order in $h\theta^{\mu\nu}$ — $a_\mu[h]$ gives no correction to the famous value — $\frac{8\pi^2}{g^2}$ — of the corresponding ordinary theory.

From eq. (3.13), one learns that, at first order in $h\theta^{\mu\nu}$, the moduli space of noncommutative SU(3) Yang-Mills theory has dimension 12 for the $k = 1$ instanton sector. Indeed, as in the ordinary case, there are 12 collective coordinates that parametrise $a_\mu[h]^{\mathrm{gen}}(x)$ in eq. (3.13): ρ, x_0^μ and the seven angles of the coset space SU(3)/ U(1).

In the next section and in appendix C, we shall use our generic first-order-in-θ-deformed instanton in the singular gauge, which we shall denote by $a_\mu^{(\mathrm{gsing})}$. $a_\mu^{(\mathrm{gsing})}$ is given by

$$\begin{aligned} a_\mu^{(\mathrm{gsing})}(x) &= U a_\mu(x)^{(\mathrm{sing})} U^\dagger, \\ a_\mu(x)^{(\mathrm{sing})}(x) &= \overline{\eta}_{a\mu\nu}\,\frac{\rho^2\,(x-x_0)_\nu}{(x-x_0)^2[(x-x_0)^2+\rho^2]}\,\tau^a \\ &\quad + \frac{2h}{\sqrt{3}}\,(\theta-\tilde{\theta})_{\mu\alpha}(x-x_0)_\alpha\,\frac{(x-x_0)^2+3\rho^2}{((x-x_0)^2+\rho^2)^3}\,T^8. \end{aligned} \tag{3.15}$$

τ^a, $a = 1, 2$ and 3, stand for the upper-left-hand corner embedding of the SU(2) generators in the generators of SU(3), both sets of generators being in their fundamental representations. $a_\mu(x)^{(\mathrm{sing})}(x)$ is obtained by applying to $a_\mu[h](x)$ in eq. (3.14) the following SU(3) gauge transformation $g(x) = \frac{i\tau_\mu^+(x-x_0)_\mu}{\sqrt{(x-x_0)^2}}$, where $\tau_\mu^+ = (\overrightarrow{\tau}, -i)$ and $\overrightarrow{\tau} = (\tau^1, \tau^2, \tau^3)$.

We shall close this section by making the connection between the first-order-in-θ-deformed instanton in eq. (3.14) and the classical vacua of noncommutative SU(3) Yang-Mills theory. In ordinary SU(3) Yang-Mills theory the instanton interpolates — along Euclidean time — between a classical vacuum in the distant past that has winding number n and and a classical vacuum in the distant future with winding number equal to $n+1$. We shall see below that the same type of phenomenon occurs when our first-order-in-θ-deformed instanton is at work. Since the phenomenon in question is most easily exhibited in the Euclidean temporal gauge, we shall perform a gauge transformation so that our first-order-in-θ-deformed instanton satisfies the gauge condition $a_4[h] = 0$. Our first-order-in-θ-deformed instanton in the temporal gauge, $a_4[h] = 0$, reads thus

$$a_i[h]^{\rm temporal}(\vec{x},\tau) = g(\vec{x},\tau)a_i[h](x)g(\vec{x},\tau)^\dagger + ig(\vec{x},\tau)\partial_i g(\vec{x},\tau)^\dagger, \quad i = 1,2,3,$$

where $x = (\vec{x},\tau)$, $a_i[h](x)$ is given by the r.h.s of eq. (3.14) and

$$g(\vec{x},\tau) = \left(e^{i\int_{-\infty}^{\tau} a_4[h](\vec{x},t)\,dt}\right)g_-(\vec{x}).$$

$a_4[h](\vec{x},t)$ is defined by the r.h.s. of eq. (3.14) and $g_-(\vec{x})\epsilon$ SU(3) is such that $g_-(|\ \vec{x}\ |\to\infty) = 1$. We see that $a_i[h]^{\rm temporal}(\tau = -\infty,\vec{x}) = ig_-(\vec{x})\partial_i g_-(\vec{x})^\dagger$ and that $a_i[h]^{\rm temporal}(\tau = +\infty,\vec{x}) = ig_+(\vec{x})\partial_i g_+(\vec{x})^\dagger$, with

$$g_+(\vec{x}) = \exp\left[\frac{-i\pi\vec{x}\vec{\sigma}}{\sqrt{\vec{x}^2+\rho^2}}\right]\,\exp[-i\pi\phi^8(\vec{x})T^8]\;g_-(\vec{x}), \qquad \phi^8(\vec{x}) = h\frac{2}{\sqrt{3}}\,\tilde{\theta}_{0i}x_i\,\frac{2\vec{x}^2+5\rho^2}{(\vec{x}^2+\rho^2)^{5/2}}.$$

Now, it can be shown — see appendix B — that, for commutative time and in the temporal gauge, all classical vacua, $a_i(\vec{x})[h]$, of the noncommutative Yang-Mills theory that admit a formal series expansion in powers of $h\theta^{\mu\nu}$ are of the form $a_i(\vec{x})[h] = ig(\vec{x})\partial_i g(\vec{x})^\dagger$ — $g(\vec{x})\epsilon$ SU(3). Now, if $a_i^-(\vec{x})$ has winding number equal to n, $a_i^+(\vec{x})$ has winding number equal to $n+1$: notice that the famous hedgehog matrix occurs in the definition of $g_+(\vec{x})$ and that $\exp(-i\pi\phi^8(\vec{x})T^8)$ has vanishing winding number. We have thus shown that for noncommutative SU(N) Yang-Mills theory, when time is commutative, our first-order-in-θ-deformed instanton field connects along Euclidean time a classical vacuum in the distant past with a classical vacuum in the distant future, the latter having a winding number which is one unit greater than the former's. This transition cannot be accomplished by continuous evolution along the classical trajectories — i.e., solutions of the equations of motion on noncommutative Minkowski space-time — since it involves a change of the winding number. The phenomenon, as in ordinary Minkowski space-time, is a genuine quantum effect: the transition is realised by tunnelling between the two vacua. We shall analyze this tunnel effect in the next section.

4. Vacuum to vacuum transition and 't Hooft vertices

For Euclidean signature and at first order in $h\theta^{\mu\nu}$, the action of non-commutative SU(3) gauge theory with n_f Dirac fermions is obtained by adding to $S_{\rm NCYM}$ in eq. (2.5) the fermionic action S_F, which is given by:

$$S_F = -\sum_{f=1}^{n_f}\int d^4x\,\overline{\psi}_f\,[\mathcal{K}[a_\mu] + im_f]\,\psi_f. \tag{4.1}$$

Here, $\mathcal{K}$ denotes the following θ-deformation of the ordinary Dirac operator $i\not{D}[a_\mu]$:

$$\mathcal{K}[a_\mu] = i\not{D}[a_\mu] - \frac{ih}{2}\theta_{\alpha\beta}\gamma_\rho f_{\rho\alpha}D_\beta[a_\mu] + \frac{ih}{8}\theta_{\alpha\beta}\gamma_\mu(\mathfrak{D}_\mu f_{\alpha\beta})[a_\mu]. \tag{4.2}$$

This operator has — at least in perturbation theory in $h\theta^{\mu\nu}$ — a discrete spectrum for gauge field configurations such as $a_\mu^{\rm (gsing)}$ in eq. (3.15). See ref. [41] for further details.

Let us denote by $\tilde{a}^a_\mu$ the quantum fluctuations around the the first-order-in-θ-deformed instanton in the singular gauge, $a_\mu^{(\mathrm{gsing})}$: $a^a_\mu = a^{a\,(\mathrm{gsing})}_\mu + \tilde{a}^a_\mu$. Then, in the first-order-in-θ-deformed instanton transition, the vacuum to vacuum amplitude for the noncommutative gauge theory with action $S = S_{\mathrm{NCYM}} + S_F$ is given, at one-loop level, by the following path integral in the background-field gauge:

$$\begin{aligned}
&\langle vac, n=1|vac, n=0\rangle \\
&\quad = e^{-\frac{8\pi^2}{g^2}} \int d\gamma\, J(\gamma) \int d\tilde{a}^a_\mu \int d\bar{c}^a\, dc^a \int \prod_f d\overline{\psi}_f\, d\psi_f \\
&\qquad e^{-\frac{1}{2}\int d^4x\; \tilde{a}^a_\mu \mathcal{M}^{ab}_{\mu\nu}[a_\rho^{(\mathrm{gsing})}]\, \tilde{a}^b_\nu + \int d^4x\; \bar{c}^a \mathcal{M}^{ab}_{gh}[a_\mu^{(\mathrm{gsing})}]\, c^a + \sum_{f=1}^{n_f} \int d^4x\, \overline{\psi}_f \left[\mathcal{K}[a_\mu^{(\mathrm{gsing})}] + im_f\right]\psi_f} \\
&\quad = e^{-\frac{8\pi^2}{g^2}} \int d\gamma\, J(\gamma) \Big(\det{}' \big(\mathcal{M}^{ab}_{\mu\nu}[a_\rho^{(\mathrm{gsing})}]\big)\Big)^{-1/2} \det\big(-\mathcal{M}^{ab}_{gh}[a_\mu^{(\mathrm{gsing})}]\big) \\
&\quad\quad \times \prod_{f=1}^{n_f} \det\big(-\mathcal{K}[a_\mu^{(\mathrm{gsing})}] - im_f\big).
\end{aligned} \tag{4.3}$$

Let us spell out now what the new symbols in the previous identity stand for. $|vac, n=0\rangle$ and $|vac, n=1\rangle$ denote vacua corresponding, respectively, to gauge field configurations with winding number $n=0$ and $n=1$, these vacua being connected by our first-order-in-θ-deformed instanton. γ denotes the collective coordinates of $a_\mu^{(\mathrm{gsing})}$, namely: its size ρ, its center x_0 and its orientation — given by U, a rigid SU(3) transformation — in the Lie algebra of SU(3). $J(\gamma)$ is the collective coordinates Jacobian, which is computed, in appendix C, from the zero modes of the operator $\mathcal{M}^{ab}_{\mu\nu}[a_\rho^{(\mathrm{gsing})}]$ defined below. The fields c^a and $\bar{c}^a$ are the ghost fields introduced in the gauge-fixing procedure. The operators $\mathcal{M}^{ab}_{\mu\nu}[a_\rho^{(\mathrm{gsing})}]$ and $\mathcal{M}^{ab}_{gh}[a_\mu^{(\mathrm{gsing})}]$ are defined by the following identities:

$$\begin{aligned}
\mathcal{M}^{ab}_{\mu\nu}[a_\sigma^{(\mathrm{gsing})}] &= \frac{\delta^2 S_{\mathrm{NCYM}}}{\delta a^a_\mu \delta a^b_\nu}\Big|_{a^a_\sigma = a^{a\,(\mathrm{gsing})}_\sigma} + \mathfrak{D}^{ac}_\mu[a_\sigma^{(\mathrm{gsing})}]\mathfrak{D}^{cb}_\nu[a_\sigma^{(\mathrm{gsing})}], \\
\mathcal{M}^{ab}_{gh}[a_\sigma^{(\mathrm{gsing})}] &= -\big(\mathfrak{D}^2[a_\sigma^{(\mathrm{gsing})}]\big)^{ab}.
\end{aligned} \tag{4.4}$$

Here, $\mathfrak{D}^{ab}_\mu[a_\rho^{(\mathrm{gsing})}] = \partial_\mu\delta^{ab} - f^{abc}a^{c\,(\mathrm{gsing})}_\mu(x)$ and S_{NCYM} is given in eq. (2.5). Let us finally note that $\det{}'\,\mathcal{M}^{ab}_{\mu\nu}[a_\rho^{(\mathrm{gsing})}]$ indicates that the zero modes of $\mathcal{M}^{ab}_{\mu\nu}[a_\rho^{(\mathrm{gsing})}]$ — see appendix C — are to be left out when computing the determinant.

Now, as shown in appendix C, $\mathcal{K}[a_\mu^{(\mathrm{gsing})}]$ has a zero mode, at least at first order in $h\theta^{\mu\nu}$. Hence,

$$\det\big(-\mathcal{K}[a_\mu^{(\mathrm{gsing})}] - im_f\big) = -im_f \prod_{\lambda>0} (-\lambda^2 - m_f^2), \tag{4.5}$$

where λ denotes a generic positive eigenvalue of $\mathcal{K}[a_\mu^{(\mathrm{gsing})}]$. To obtain eq. (4.5), we have taken into account that the non-zero eigenvalues of $\mathcal{K}[a_\mu^{(\mathrm{gsing})}]$ come in pairs $(\lambda, -\lambda)$. The spectrum of $\mathcal{K}[a_\mu^{(\mathrm{gsing})}]$ is discrete, at least in the perturbative expansion in $h\theta^{\mu\nu}$, due to the fast fall-off of $a_\mu^{(\mathrm{gsing})}$ at infinity.

That the vacuum to vacuum amplitude in eq. (4.3) vanishes, or nearly vanishes, when massless, or nearly massless, quarks couple to the first-order-in-θ-deformed instanton can

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be seen as a consequence of the $\mathrm{U}(1)_A$ anomaly. Indeed, using the results in refs. [31, 41], one concludes that in the first-order-in-θ-deformed instanton transition the chiral charge associated to each massless flavour, f, changes compulsorily by two units: the first-order-in-θ-deformed instanton turns a left handed quark into a right handed quark. This selection rule would be broken by a non-zero $\langle vac, n = 1|vac, n = 0\rangle$. Since the first-order-in-θ-deformed instanton turns a left-handed quark of massless flavour into a right-handed quark with the same flavour, to obtain non-zero amplitudes one must insert enough pairs of quark-anti-quark fields between $|vac, n = 0\rangle$ and $\langle vac, n = 1|$. Indeed, the quark propagator of the flavour f in the first-order-in-θ-deformed instanton reads

$$\langle\psi_f(x)\overline{\psi}(y)_f\rangle^{(\theta\text{definst})} = -\frac{\psi_0(x)\psi_0(y)^\dagger}{im_f} - \sum_{\lambda\neq 0}\frac{\psi_\lambda(x)\psi^\dagger_\lambda(y)}{\lambda + im_f}, \tag{4.6}$$

where $\psi_0(x)$ stands for the zero-mode of $\mathcal{K}[a_\mu^{(\text{gsing})}]$ worked out in appendix C and $\psi_\lambda(x)$ denotes generically the remaining eigenfunctions of this operator. Suppose now that the masses of the n_f quark flavours are taken to zero. Then, in this chiral limit, the Green function $\langle\prod_{f=1}^{n_f}\psi_f(x)\overline{\psi}(y)_f\rangle^{(\theta\text{definst})}$ has a non-vanishing value, for the pole at $m_f = 0$ of the propagator in eq. (4.6) cancels the linear contribution in m_f to the determinant in eq. (4.5).

As in the ordinary case [38], the coupling between left-handed and right-handed massless quarks through the first-order-in-θ-deformed instanton can be mimicked by using an effective Lagrangian. This coupling does not occur at any order in the perturbative expansion in powers of the coupling constant and the effective Lagrangian, $\mathcal{L}_{\text{eff}} = \sum_{n=0}^{n_f} L_{2n}$, that simulates it is a sum of non-local interactions — called 't Hooft vertices —, each involving $2n$ fermions. In these non-local interactions quarks are emitted or absorbed in the zero-mode wave function $\psi_0(x)$. The contribution L_{2n} matches, as $m_f \to 0$ — $f = 1,\ldots,n_f$ —, the leading contribution to the amputated Green function obtained from $\langle\prod_{f'=1}^{n}\psi_{f'}(x_{f'})\overline{\psi}(y_{f'})_{f'}\rangle^{(\theta\text{definst})}$. The amputation is to be carried out with the Dirac free propagator, and $\{\psi_{f'}(x)\}$ stands for any set of n — with $n \leq n_f$ — fermion fields. Now, it is further assumed that the previous Green function is normalized to the vacuum to vacuum amplitude in the perturbation theory background $a_\mu = 0$. Below we shall work out this effective Lagrangian for one, two and three nearly massless flavours — i.e., $n_f = 1, 2$ and 3 — to obtain the first order in θ corrections to the ordinary results obtained in refs. [43, 44, 2].

4.1 One light flavour

In this case we need to compute the leading contribution as, say, $m_1 = m \to 0$, to $\langle vac, n = 1|vac, n = 0\rangle/\langle vac, n = 0|vac, n = 0\rangle$ and $\langle\psi(x)\psi^\dagger(y)\rangle^{(\theta\text{definst})}/\langle vac, n = 0|vac, n = 0\rangle$. $\psi(x)$ denotes the field of the light quark. It turns out that the contribution to $\langle vac, n = 1|vac, n = 0\rangle/\langle vac, n = 0|vac, n = 0\rangle$ which is linear in $\theta^{\mu\nu}$ vanishes since it must be proportional to $\theta_{\mu\nu}g_{\mu\nu}$, $g_{\mu\nu}$ being the space-time metric. Hence,

$$\frac{\langle vac, n = 1|vac, n = 0\rangle^{(\theta\text{definst})}}{\langle vac, n = 0|vac, n = 0\rangle} \approx \int\frac{d\rho\, d^4x_0}{\rho^5}\, d_0^{(n_f=1)}(\rho)\, m\rho,$$

where $d_0^{(n_f=1)}(\rho)$ is the ordinary "reduced" instanton density [45]:

$$d_0^{(n_f)}(\rho) = C^{(n_f)} \left(\frac{8\pi^2}{g^2(\rho)}\right)^6 e^{-\left(\frac{8\pi^2}{g^2(\rho)}\right)}.$$

C_{n_f} is constant which depends on the number of light flavours and the regularization scheme.

Unlike $\langle vac, n = 1|vac, n = 0\rangle/\langle vac, n = 0|vac, n = 0\rangle$, $\langle\psi(x)\overline{\psi}(y)\rangle^{(\theta\text{definst})}/\langle vac, n = 0|vac, n = 0\rangle$ receives contributions which are linear in $\theta^{\mu\nu}$. Indeed, $m\rho \ll 1$ leads to

$$\begin{aligned}
&\frac{\langle\psi(x)\psi^\dagger(y)\rangle^{(\theta\text{definst})}}{\langle vac, n = 0|vac, n = 0\rangle}\\
&\quad\approx \int \frac{d\rho d^4x_0 dU}{\rho^5}\, d_0^{(n_f=1)}(\rho)\, m\rho\, \frac{\psi_0(x-x_0)\psi_0^\dagger(y-x_0)}{-im}\\
&\quad= \int \frac{d\rho d^4x_0 dU}{-i\rho^4}\, d_0^{(n_f=1)}(\rho)\Big[\psi_0^{(0)}(x-x_0)\psi_0^{(0)\,\dagger}(y-x_0)+ h\psi_0^{(0)}(x-x_0)\psi_0^{(1a)\,\dagger}(y-x_0)\\
&\qquad\qquad + h\psi_0^{(1a)}(x-x_0)\psi_0^{(0)\,\dagger}(y-x_0)+h\psi_0^{(0)}(x-x_0)\psi_0^{(1b)\,\dagger}(y-x_0)\\
&\qquad\qquad + h\psi_0^{(1b)}(x-x_0)\psi_0^{(0)\,\dagger}(y-x_0) + O(h^2\theta^2)\Big],
\end{aligned}$$

where $\psi_0^{(0)}(x)$ is the zero mode of the ordinary Dirac operator in the ordinary instanton field, and $h\psi_0^{(1a)}(x)$ and $h\psi^{(1b)}$ are the corrections of order $h\theta^{\mu\nu}$ to $\psi_0^{(0)}(x)$ that make, at first order in $h\theta^{\mu\nu}$, $\psi_0(x) = \psi_0^{(0)}(x) + h\psi_0^{(1a)}(x) + h\psi_0^{(1b)}(x)$ the zero mode of the operator $\mathcal{K}[a_\mu^{(\text{gsing})}]$. See appendix C, for definitions and further details. Since SU(3) is compact, and following ref. [44], the averaging over SU(3) first-order-in-θ-deformed instanton orientations is carried out by using the first two SU(3) integrals in eq. (E.2) of appendix E.

Let τ^a, $a = 1, 2$ and 3 denote the upper-left-hand corner embedding of the SU(2) generators in the generators of SU(3) in the fundamental representation. We define $\tau_\mu^\pm = (\vec{\tau}, \mp i)$, $\tau_{\mu\nu} = \frac{1}{4i}(\tau_\mu^-\tau_\nu^+ - \tau_\mu^-\tau_\nu^+)$. Then, taking into account the definitions in eqs. (C.11), (C.12) and (C.13) of appendix C, and using the conventions in appendix E, one obtains the following expressions:

$$\begin{aligned}
\psi_{0\,im}^{(0)}(x)\psi_{0\,jn}^{(0)\,\dagger}(y) &= \frac{1}{8}\,\phi(x)\phi(y)[(\not{x} - \not{x}_0)\gamma_\mu\gamma_\nu(\not{y} - \not{x}_0)P_R]_{ij}[U\tau_\mu^-\tau_\nu^+U^\dagger]_{mn},\\
\psi_{0\,im}^{(0)}(x)\psi_{0\,jn}^{(1a)\,\dagger}(y) &= \frac{1}{8}\phi(x)(\Gamma_{\rho\sigma}(y)(y-x_0)_\rho(y-x_0)_\alpha+\Lambda_{\alpha\sigma}(y))[(\not{x} - \not{x}_0)\gamma_\mu\gamma_\nu(\not{y} - \not{x}_0)P_R]_{ij}\\
&\quad\times[U\tau_\mu^-\tau_\nu^+\tau_{\sigma\alpha}U^\dagger]_{mn},\\
\psi_{0\,im}^{(0)}(x)\psi_{0\,jn}^{(1b)\,\dagger}(y) &= \frac{1}{8}\,\phi(x)\chi^*_{\alpha\sigma}(y)(y-x_0)_\alpha[(\not{x} - \not{x}_0)\gamma_\mu\gamma_\nu\gamma_\sigma P_R]_{ij}[U\tau_\mu^-\tau_\nu^+U^\dagger]_{mn},
\end{aligned}$$

from which one concludes that the effective Lagrangian $\mathcal{L}_{\text{eff}}^{(n_f=1)}(x)$ is given by the following equations:

$$\begin{aligned}
\int d^4x\ \mathcal{L}_{\text{eff}}^{(n_f=1)}(x) &= \int d^4x\ \mathfrak{L}_0(x) + \int d^4x\ \mathfrak{L}_2(x),\\
\mathfrak{L}_0(x) &= \int \frac{d\rho}{\rho^5}\, d_0^{(n_f=1)}(\rho)\, m\rho,\ \ \mathfrak{L}_2(x) = \int \frac{d\rho}{\rho^5}\, d_0^{(n_f=1)}(\rho) \int \frac{d^4p}{(2\pi)^4}\, e^{-ipx}\, \mathcal{Y}_2(p),
\end{aligned}$$

$$
\begin{aligned}
\mathcal{Y}_2(p) &= \mathcal{Y}_2^{(0)}(p) + h\,\mathcal{Y}_2^{(1)}(p),\\
\mathcal{Y}_2^{(0)}(p) &= \frac{i}{3}\,\rho\,\overline{\mathcal{Q}}_R(p)\mathcal{Q}_L(p),\\
\mathcal{Y}_2^{(1)}(p) &= \frac{i}{3}\,\rho\Big[\overline{\mathcal{S}}_R(p)\mathcal{Q}_L(p) - \overline{\mathcal{Q}}_R(p)\mathcal{S}_L(p)\\
&\qquad - \overline{\mathcal{R}}_{\alpha\sigma,R}(p)\gamma_{\sigma\alpha}\mathcal{Q}_L(p) - \overline{\mathcal{Q}}_R(p)\gamma_{\sigma\alpha}\mathcal{R}_{\alpha\sigma,L}(p)\Big],
\end{aligned}
\tag{4.7}
$$

where

$$
\begin{aligned}
\mathcal{Q}(p) &\equiv \phi'(u)u\,\psi(p), & \overline{\mathcal{Q}}(p) &\equiv \phi'(u)u\,\overline{\psi}(p),\\
\mathcal{R}_{\alpha\sigma}(p) &\equiv (-\partial_{\rho\alpha}\Gamma'_{\rho\sigma}(u) + \Lambda'_{\alpha\sigma}(u))u\,\psi(p), & \overline{\mathcal{R}}_{\alpha\sigma}(p) &\equiv (-\partial_{\rho\alpha}\Gamma'_{\rho\sigma}(u) + \Lambda'_{\alpha\sigma}(u))u\,\overline{\psi}(p),\\
\mathcal{S}(p) &\equiv \frac{\chi'_{\alpha\sigma}(u)p_\alpha}{u}\gamma_\sigma \not{p}\,\psi(p), & \overline{\mathcal{S}}(p) &\equiv \frac{\chi'_{\alpha\sigma}(u)p_\alpha}{u}\,\overline{\psi}(p)\not{p}\gamma_\sigma,
\end{aligned}
$$

with $u = \sqrt{p^2}$. Derivatives with respect to u are denoted by the super-script $'$. In eq. (E.3) in appendix E, the functions $\phi(u)$, $\Gamma_{\rho\sigma}(u)$, $\Lambda_{\alpha\sigma}(u)$ and $\chi_{\alpha\sigma}(u)$ are given in terms of modified Bessel functions. $\mathcal{Y}_2^{(0)}(p)$ is the ordinary result — see refs. [1, 2] — and $\mathcal{Y}_2^{(1)}(p)$ is the first order noncommutative correction. Note that neither $\mathcal{Y}_2^{(0)}(p)$ nor $\mathcal{Y}_2^{(1)}(p)$ are invariant under chiral transformations. This shows that the classical chiral symmetry of the massless theory is broken in the quantum theory.

One expects that the previous effective Lagrangian gives right Physics in the low energy regime: $p\rho \ll 1$, $h\theta^{\mu\nu}\rho^{-2} \ll 1$. Using the low-momentum approximations in appendix E, one obtains the following low-energy expressions for the two-field contribution to $\mathcal{L}_{\rm eff}^{(n_f=1)}$:

$$
\mathfrak{L}_2(p) = i\int \frac{d\rho}{\rho^5}\, d_0^{(n_f=1)}(\rho)\,\frac{4\pi^2\rho^3}{3}\overline{\psi}_R(p)\big[1 + h\mathcal{T}\big]\psi_L(p).
$$

Here, $\mathcal{T} = -\frac{4}{3p^2\rho^2}(\theta - \tilde{\theta})_{\mu\nu}\gamma_{\alpha\nu}p_\mu p_\alpha$, with $\gamma_{\alpha\nu} = \frac{1}{4i}[\gamma_\alpha, \gamma_\nu]$. Notice that the ordinary contribution to $\mathcal{L}_2(p)$ just above acts like a mass term. This interpretation is spoiled by the first order corrections in $h\theta^{\mu\nu}$.

4.2 Two light flavours

Now, in eq. (4.1), $n_f = 2$ and $m_f\rho \ll 1$, $f = 1, 2$. The effective Lagrangian, $\mathcal{L}_{\rm eff}^{(n_f=2)}(x)$, that yields the $m_f \to 0$ leading contributions to

$$
\frac{\langle vac, n=1|vac, n=0\rangle}{\langle vac, n=0|vac, n=0\rangle},\ \frac{\langle \psi_f(x)\psi_f^\dagger(y)\rangle^{(\theta\text{definst})}}{\langle vac, n=0|vac, n=0\rangle}\ \text{and}\ \frac{\langle \prod_{f=1,2}\psi_f(x_f)\psi_f^\dagger(y_f)\rangle^{(\theta\text{definst})}}{\langle vac, n=0|vac, n=0\rangle},
$$

reads

$$
\begin{aligned}
\int d^4x\ \mathcal{L}_{\rm eff}^{n_f=2}(x) &= \int d^4x\ \int \frac{d\rho}{\rho^5}\, d_0^{(n_f=2)}(\rho)\prod_{f=1,2}\,[m_f\rho + \mathcal{Y}_2(x,\psi_f)]\\
&\quad + \int \prod_{j=1,2}\frac{dp_j}{(2\pi)^4}\frac{dq_j}{(2\pi)^4}\delta\left(\sum_{j=1,2}p_j - \sum_{j=1,2}q_j\right)\mathfrak{L}_4(p_1,p_2,q_1,q_2).
\end{aligned}
\tag{4.8}
$$

Here, $\mathcal{Y}_2(x,\psi_f)$ is obtained from the corresponding expression in eq. (4.7) by performing the Fourier transform and then applying it to the fermion ψ_f with light mass m_f, and $\mathfrak{L}_4(p_1,p_2,q_1,q_2)$ is given by the following identities:

$$
\begin{aligned}
\mathfrak{L}_4(p_1,p_2,q_1,q_2) &= \mathfrak{L}_4^{(0)}(p_1,p_2,q_1,q_2)+h\mathfrak{L}_4^{(1)}(p_1,p_2,q_1,q_2),\\
\mathfrak{L}_4^{(0)}(p_1,p_2,q_1,q_2) &= \int\frac{d\rho}{\rho^5}\,d_0^{(n_f=2)}(\rho)\,\mathcal{Y}_4^{(0)}(p_1,p_2,q_1,q_2),\\
\mathfrak{L}_4^{(1)}(p_1,p_2;q_1,q_2) &= \int\frac{d\rho}{\rho^5}\,d_0^{(n_f=1)}(\rho)\,\mathcal{Y}_4^{(1)}(p_1,p_2,q_1,q_2),\\
\mathcal{Y}_4^{(0)}(p_1,p_2,q_1,q_2) &= \frac{-\rho^2}{32}\Big\{\frac{1}{3}(\overline{\mathcal{Q}}_R^1(p_1)\lambda^a\mathcal{Q}_L^1(q_1))(\overline{\mathcal{Q}}_R^2(p_2)\lambda^a\mathcal{Q}_L^2(q_2))\\
&\qquad +(\overline{\mathcal{Q}}_R^1(p_1)\gamma_{\mu\nu}\lambda^a\mathcal{Q}_L^1(q_1))(\overline{\mathcal{Q}}_R^2(p_2)\gamma_{\mu\nu}\lambda^a\mathcal{Q}_L^2(q_2))\Big\},\\
\mathcal{Y}_4^{(1)}(p_1,p_2,q_1,q_2) &= \frac{-\rho^2}{32}\Big\{\frac{1}{3}(\overline{\mathcal{S}}_R^1(p_1)\lambda^a\mathcal{Q}_L^1(q_1)-\overline{\mathcal{Q}}_R^1(p_1)\lambda^a\mathcal{S}_L^1(q_1))(\overline{\mathcal{Q}}_R^2(p_2)\lambda^a\mathcal{Q}_L^2(q_2))\\
&\qquad +(\overline{\mathcal{S}}_R^1(p_1)\gamma_{\mu\nu}\lambda^a\mathcal{Q}_L^1(q_1)-\overline{\mathcal{Q}}_R^1(p_1)\gamma_{\mu\nu}\lambda^a\mathcal{S}_L^1(q_1))(\overline{\mathcal{Q}}_R^2(p_2)\gamma_{\mu\nu}\lambda^a\mathcal{Q}_L^2(q_2))\Big\}\\
&\quad +\frac{\rho^2}{32}\Big\{\frac{1}{3}(\overline{\mathcal{Q}}_R^1(p_1)\gamma_{\sigma\alpha}\lambda^a\mathcal{R}_{\alpha\sigma,L}^1(q_1)+\overline{\mathcal{R}}_{\alpha\sigma,R}^1(p_1)\gamma_{\sigma\alpha}\lambda^a\mathcal{Q}_L^1(q_1))(\overline{\mathcal{Q}}_R^2(p_2)\lambda^a\mathcal{Q}_L^2(q_2))\\
&\qquad +(\overline{\mathcal{Q}}_R^1(p_1)\lambda^a\mathcal{R}_{\alpha\sigma,L}^1(q_1)+\overline{\mathcal{R}}_{\alpha\sigma,R}^1(p_1)\lambda^a\mathcal{Q}_L^1(q_1))(\overline{\mathcal{Q}}_R^2(p_2)\gamma_{\sigma\alpha}\lambda^a\mathcal{Q}_L^2(q_2))\\
&\qquad +i\big[(\overline{\mathcal{Q}}_R^1(p_1)\gamma_{\alpha\nu}\lambda^a\mathcal{R}_{\alpha\sigma,L}^1(q_1)-\overline{\mathcal{R}}_{\alpha\sigma,R}^1(p_1)\gamma_{\alpha\nu}\lambda^a\mathcal{Q}_L^1(q_1))\\
&\qquad\quad\times(\overline{\mathcal{Q}}_R^2(p_2)\gamma_{\nu\sigma}\lambda^a\mathcal{Q}_L^2(q_2))-(\alpha\leftrightarrow\sigma)\big]\Big\}+(1\leftrightarrow 2).
\end{aligned}
\tag{4.9}
$$

$\mathcal{Q}^f(p)$, $\overline{\mathcal{Q}}^f(p)$, $\mathcal{R}^f_{\alpha\sigma}(p)$, $\overline{\mathcal{R}}^f_{\alpha\sigma}(p)$, $\mathcal{S}^f(p)$ and $\overline{\mathcal{S}}^f(p)$ in the previous equation are defined by the following equations:

$$
\begin{aligned}
\mathcal{Q}^f(p) &\equiv \phi'(u)u\,\psi_f(p), & \overline{\mathcal{Q}}^f(p) &\equiv \phi'(u)u\,\overline{\psi}_f(p),\\
\mathcal{R}^f_{\alpha\sigma}(p) &\equiv (-\partial_{\rho\alpha}\Gamma'_{\rho\sigma}(u)+\Lambda'_{\alpha\sigma}(u))u\,\psi_f(p), & \overline{\mathcal{R}}^f_{\alpha\sigma}(p) &\equiv (-\partial_{\rho\alpha}\Gamma'_{\rho\sigma}(u)+\Lambda'_{\alpha\sigma}(u))u\,\overline{\psi}_f(p),\\
\mathcal{S}^f(p) &\equiv \frac{\chi'_{\alpha\sigma}(u)p_\alpha}{u}\gamma_\sigma \not{p}\,\psi_f(p), & \overline{\mathcal{S}}^f(p) &\equiv \frac{\chi'_{\alpha\sigma}(u)p_\alpha}{u}\,\overline{\psi}_f(p)\not{p}\gamma_\sigma,
\end{aligned}
$$

where $u=\sqrt{p^2}$ and the derivatives with respect to u are denoted by the super-script $'$. The functions $\phi(u)$, $\Lambda_{\alpha\sigma}(u)$, $\Lambda_{\alpha\sigma}(u)$ and $\chi_{\alpha\sigma}(u)$ are given in terms of modified Bessel functions in eq. (E.3) of appendix E.

To obtain $\mathcal{L}_{\text{eff}}^{(n_f=2)}(x)$ in eq. (4.8), averages over the SU(3) are to be carried out with the help of the first three equalities in eq. (E.2) of appendix E. Notice that if we set $h=0$ in eq. (4.8) the ordinary 't Hooft vertex for two light flavours [1, 2] is recovered.

As in the one-flavour case, we expect that $\mathcal{L}_{\text{eff}}^{(n_f=2)}(x)$ in eq. (4.8) gives the correct Physics in the low momenta limit: $p_i\rho \ll 1, q_i\rho \ll 1$, $i=1,2$. In this limit, $\mathfrak{L}_4(p_1,p_2,q_1,q_2)$ boils down to

$$\mathfrak{L}_4(p_1,p_2,q_1,q_2)$$

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$$
\begin{aligned}
&\approx -\int \frac{d\rho}{\rho^5}\, d_0^{(n_f=2)}(\rho)\Bigg[\frac{3}{32}\left(\frac{4\pi^2\rho^3}{3}\right)^2 \{(\overline{\psi}_{1\,R}(p_1)\lambda^a\psi_{1\,L}(q_1))(\overline{\psi}_{2\,R}(p_2)\lambda^a\psi_{2\,L}(q_2)) \\
&\qquad\qquad + 3(\overline{\psi}_{1\,R}(p_1)\gamma_{\mu\nu}\lambda^a\psi_{1\,L}(q_1))(\overline{\psi}_{2\,R}(p_2)\gamma_{\mu\nu}\lambda^a\psi_{2\,L}(q_2))\}\Bigg] \\
&\quad -h\int \frac{d\rho}{\rho^5}\, d_0^{(n_f=2)}(\rho)\Bigg[\frac{3}{32}\left(\frac{4\pi^2\rho^3}{3}\right)^2 \{(\overline{\psi}_{1\,R}(p_1)[\overline{\mathcal{O}}(p_1)-\mathcal{O}(q_1)]\lambda^a\psi_{1\,L}(q_1))(\overline{\psi}_{2\,R}(p_2)\lambda^a\psi_{2\,L}(q_2)) \\
&\qquad\qquad + 3(\overline{\psi}_{1\,R}(p_1)[\overline{\mathcal{O}}(p_1)\gamma_{\mu\nu}-\gamma_{\mu\nu}\mathcal{O}(q_1)]\lambda^a\psi_{1\,L}(q_1))(\overline{\psi}_{2\,R}(p_2)\gamma_{\mu\nu}\lambda^a\psi_{2\,L}(q_2))\} \\
&\quad +(1\leftrightarrow 2)\Bigg],
\end{aligned}
$$

where

$$\overline{\mathcal{O}}(p) = \frac{i}{3\rho^2 p^2}\,(\theta-\tilde{\theta})_{\mu\nu}\not{p}\,p_\mu\gamma_\nu \quad \text{and} \quad \mathcal{O}(p) = \frac{i}{3\rho^2 p^2}\,(\theta-\tilde{\theta})_{\mu\nu}\gamma_\nu\not{p}\,p_\mu. \tag{4.10}$$

To obtain eq. (4.10), we have used the approximations in eq. (E.4) of appendix E. Of course, if the contributions linear in $\theta^{\mu\nu}$ are dropped one obtains the ordinary contributions in ref. [43].

4.3 Three light flavours

We shall finally give the effective Lagrangian, $\mathcal{L}_{\rm eff}^{(n_f=3)}(x)$, for the case of three light fermions: the fermionic action is the action in eq. (4.1) for $n_f = 3$.

Let us first introduce some notation. $\mathcal{Y}_4(x,\psi_f,\psi_{f'})$ is defined by

$$
\begin{aligned}
\mathcal{Y}_4(x,\psi_f,\psi_{f'}) = \int \prod_{j=1}^{2} \frac{dp_j}{(2\pi)^4}\frac{dq_j}{(2\pi)^4}\, e^{-i(\sum_{j=1}^2 p_i - \sum_{j=1}^2 q_i)x} \\
\times\Big[\mathcal{Y}_4^{(0)}(p_1,p_2,q_1,q_2) + h\mathcal{Y}_4^{(1)}(p_1,p_2,q_1,q_2)\Big],
\end{aligned}
$$

once ψ_1 and ψ_2 are replaced with ψ_f and $\psi_{f'}$, respectively, both in $\mathcal{Y}_4^{(0)}(p_1,p_2,q_1,q_2)$ and $\mathcal{Y}_4^{(1)}(p_1,p_2,q_1,q_2)$. $\mathcal{Y}_4^{(0)}(p_1,p_2,q_1,q_2)$ and $\mathcal{Y}_4^{(1)}(p_1,p_2,q_1,q_2)$ are given in eq. (4.9). $\mathcal{Y}_2(x,\psi_f)$ shall denote the same quantity that occurred in eq. (4.8). Then, $\mathcal{L}_{\rm eff}^{(n_f=3)}(x)$ is given by the expressions that follow:

$$
\begin{aligned}
\int d^4x\; \mathcal{L}_{\rm eff}^{(n_f=3)}(x) &= \int d^4x \int \frac{d\rho}{\rho^5}\, d_0^{(n_f=3)}(\rho) \prod_{f=1,2,3} [m_f\rho + \mathcal{Y}_2(x,\psi_f)] \\
&\quad + \int d^4x \int \frac{d\rho}{\rho^5}\, d_0^{(n_f=3)}(\rho)\Big\{[m_1\rho + \mathcal{Y}_2(x,\psi_1)]\mathcal{Y}_4(x,\psi_2,\psi_3) \\
&\qquad\qquad + [m_2\rho + \mathcal{Y}_2(x,\psi_2)]\mathcal{Y}_4(x,\psi_1,\psi_3) \\
&\qquad\qquad + [m_3\rho + \mathcal{Y}_2(x,\psi_3)]\mathcal{Y}_4(x,\psi_1,\psi_2)\Big\} \\
&\quad + \int \prod_{j=1}^{3} \frac{dp_j}{(2\pi)^4}\frac{dq_j}{(2\pi)^4}\,\delta\left(\sum_{j=1}^3 p_j - \sum_{j=1}^3 q_j\right)\mathfrak{L}_6(p_1,p_2,p_3,q_1,q_2,q_3),
\end{aligned}
$$

where

$$\mathfrak{L}_6(p_1,p_2,p_3,q_1,q_2,q_3) = \mathfrak{L}_6^{(0)}(p_1,p_2,p_3,q_1,q_2,q_3) + h\mathfrak{L}_6^{(1)}(p_1,p_2,p_3,q_1,q_2,q_3),$$

$$
\begin{aligned}
&\mathfrak{L}_6^{(0)}(p_1,p_2,p_3,q_1,q_2,q_3) \;=\; \int \frac{d\rho}{\rho^5}\, d_0^{(n_f=3)}(\rho)\Bigg[\\
&\quad \frac{-i\rho^3}{64}\Bigg\{ d_{abc}\Bigg[-\frac{1}{15}(\overline{\mathcal{Q}}^1_R(p_1)\lambda^a\mathcal{Q}^1_L(q_1))(\overline{\mathcal{Q}}^2_R(p_2)\lambda^b\mathcal{Q}^2_L(q_2))(\overline{\mathcal{Q}}^3_R(p_3)\lambda^c\mathcal{Q}^3_L(q_3))\\
&\quad +\left(\frac{1}{5}(\overline{\mathcal{Q}}^1_R(p_1)\gamma_{\mu\nu}\lambda^a\mathcal{Q}^1_L(q_1))(\overline{\mathcal{Q}}^2_R(p_2)\gamma_{\mu\nu}\lambda^b\mathcal{Q}^2_L(q_2))(\overline{\mathcal{Q}}^3_R(p_3)\lambda^c\mathcal{Q}^3_L(q_3))+(3\leftrightarrow 1)+(3\leftrightarrow 2)\right)\Bigg]\\
&\quad -\frac{2}{3}f_{abc}(\overline{\mathcal{Q}}^1_R(p_1)\gamma_{\alpha\beta}\lambda^a\mathcal{Q}^1_L(q_1))(\overline{\mathcal{Q}}^2_R(p_2)\gamma_{\beta\delta}\lambda^b\mathcal{Q}^2_L(q_2))(\overline{\mathcal{Q}}^3_R(p_3)\gamma_{\delta\alpha}\lambda^c\mathcal{Q}^3_L(q_3))\Bigg\}\Bigg],\\
&\mathfrak{L}_6^{(1)}(p_1,p_2,p_3,q_1,q_2,q_3) \;=\; \int \frac{d\rho}{\rho^5}\, d_0^{(n_f=3)}(\rho)\Bigg[\\
&\quad \frac{-i\rho^3}{64}\Bigg\{ d_{abc}\Bigg[-\frac{1}{15}(\overline{\mathcal{S}}^1_R(p_1)\lambda^a\mathcal{Q}^1_L(q_1)-\overline{\mathcal{Q}}^1_R(p_1)\lambda^a\mathcal{S}^1_L(q_1))(\overline{\mathcal{Q}}^2_R(p_2)\lambda^b\mathcal{Q}^2_L(q_2))(\overline{\mathcal{Q}}^3_R(p_3)\lambda^c\mathcal{Q}^3_L(q_3))\\
&\quad +\frac{1}{5}\Big((\overline{\mathcal{S}}^1_R(p_1)\gamma_{\mu\nu}\lambda^a\mathcal{Q}^1_L(q_1)-\overline{\mathcal{Q}}^1_R(p_1)\gamma_{\mu\nu}\lambda^a\mathcal{S}^1_L(q_1))(\overline{\mathcal{Q}}^2_R(p_2)\gamma_{\mu\nu}\lambda^b\mathcal{Q}^2_L(q_2))(\overline{\mathcal{Q}}^3_R(p_3)\lambda^c\mathcal{Q}^3_L(q_3))\\
&\quad +(3\leftrightarrow 2)+(\overline{\mathcal{S}}^1_R(p_1)\lambda^a\mathcal{Q}^1_L(q_1)-\overline{\mathcal{Q}}^1_R(p_1)\lambda^a\mathcal{S}^1_L(q_1))(\overline{\mathcal{Q}}^2_R(p_2)\gamma_{\mu\nu}\lambda^b\mathcal{Q}^2_L(q_2))(\overline{\mathcal{Q}}^3_R(p_3)\gamma_{\mu\nu}\lambda^c\mathcal{Q}^3_L(q_3))\Big)\Bigg]\\
&\quad -\frac{2}{3}f_{abc}(\overline{\mathcal{S}}^1_R(p_1)\gamma_{\alpha\beta}\lambda^a\mathcal{Q}^1_L(q_1)-\overline{\mathcal{Q}}^1_R(p_1)\gamma_{\alpha\beta}\lambda^a\mathcal{S}^1_L(q_1))(\overline{\mathcal{Q}}^2_R(p_2)\gamma_{\beta\delta}\lambda^b\mathcal{Q}^2_L(q_2))(\overline{\mathcal{Q}}^3_R(p_3)\gamma_{\delta\alpha}\lambda^c\mathcal{Q}^3_L(q_3))\\
&\quad \Bigg\}-\frac{i\rho^3}{128}\Bigg\{\frac{2}{5}d_{abc}\Bigg[\\
&\quad \frac{1}{3}(\overline{\mathcal{Q}}^1_R(p_1)\gamma_{\sigma\alpha}\lambda^a\mathcal{R}^1_{\alpha\sigma,L}(q_1)+\overline{\mathcal{R}}^1_{\alpha\sigma,R}(p_1)\gamma_{\sigma\alpha}\lambda^a\mathcal{Q}^1_L(q_1))(\overline{\mathcal{Q}}^2_R(p_2)\lambda^b\mathcal{Q}^2_L(q_2))(\overline{\mathcal{Q}}^3_R(p_3)\lambda^c\mathcal{Q}^3_L(q_3))\\
&\quad -\big((\overline{\mathcal{Q}}^1_R(p_1)\lambda^a\mathcal{R}^1_{\alpha\sigma,L}(q_1)+\overline{\mathcal{R}}^1_{\alpha\sigma,R}(p_1)\lambda^a\mathcal{Q}^1_L(q_1))(\overline{\mathcal{Q}}^2_R(p_2)\lambda^b\mathcal{Q}^2_L(q_2))(\overline{\mathcal{Q}}^3_R(p_3)\gamma_{\sigma\alpha}\lambda^c\mathcal{Q}^3_L(q_3))\\
&\quad +(2\leftrightarrow 3)\big)-(\overline{\mathcal{Q}}^1_R(p_1)\gamma_{\sigma\alpha}\lambda^a\mathcal{R}^1_{\alpha\sigma,L}(q_1)+\overline{\mathcal{R}}^1_{\alpha\sigma,R}(p_1)\gamma_{\sigma\alpha}\lambda^a\mathcal{Q}^1_L(q_1))(\overline{\mathcal{Q}}^2_R(p_2)\gamma_{\mu\nu}\lambda^b\mathcal{Q}^2_L(q_2))\cdot\\
&\quad \cdot(\overline{\mathcal{Q}}^3_R(p_3)\gamma_{\mu\nu}\lambda^c\mathcal{Q}^3_L(q_3))+i\big((\overline{\mathcal{Q}}^1_R(p_1)\gamma_{\sigma\nu}\lambda^a\mathcal{R}^1_{\alpha\sigma,L}(q_1)-\overline{\mathcal{R}}^1_{\alpha\sigma,R}(p_1)\gamma_{\sigma\nu}\lambda^a\mathcal{Q}^1_L(q_1))\cdot\\
&\quad \cdot(\overline{\mathcal{Q}}^2_R(p_2)\gamma_{\nu\alpha}\lambda^b\mathcal{Q}^2_L(q_2))(\overline{\mathcal{Q}}^3_R(p_3)\lambda^c\mathcal{Q}^3_L(q_3))+(2\leftrightarrow 3)-(\alpha\leftrightarrow\sigma)\big)\Bigg]+\frac{1}{3}f_{abc}\Bigg[\\
&\quad \big((\overline{\mathcal{Q}}^1_R(p_1)\lambda^a\mathcal{R}^1_{\alpha\sigma,L}(q_1)+\overline{\mathcal{R}}^1_{\alpha\sigma,R}(p_1)\lambda^a\mathcal{Q}^1_L(q_1))(\overline{\mathcal{Q}}^2_R(p_2)\gamma_{\alpha\nu}\lambda^b\mathcal{Q}^2_L(q_2))(\overline{\mathcal{Q}}^3_R(p_3)\gamma_{\nu\sigma}\lambda^c\mathcal{Q}^3_L(q_3))\\
&\quad -(\alpha\leftrightarrow\sigma)\big)-2i\big((\overline{\mathcal{Q}}^1_R(p_1)\gamma_{\mu\nu}\lambda^a\mathcal{R}^1_{\alpha\sigma,L}(q_1)-\overline{\mathcal{R}}^1_{\alpha\sigma,R}(p_1)\gamma_{\mu\nu}\lambda^a\mathcal{Q}^1_L(q_1))(\overline{\mathcal{Q}}^2_R(p_2)\gamma_{\mu\nu}\lambda^b\mathcal{Q}^2_L(q_2))\\
&\quad (\overline{\mathcal{Q}}^3_R(p_3)\gamma_{\sigma\alpha}\lambda^c\mathcal{Q}^3_L(q_3))-(2\leftrightarrow 3)\big)\Big]\Big\}+(1\leftrightarrow 2)+(1\leftrightarrow 3)\Bigg].
\end{aligned}
$$

To obtain the average over the SU(3) orientations leading to the previous equation, we have employed the last integral in eq. (E.2) of appendix E; some values of the structure constants of SU(3) were substituted. The low momenta — i.e., $p_i\rho \ll 1, q_i\rho \ll 1$, $i = 1,2,3$ — approximation to $\mathcal{L}_6(p_1,p_2,p_3,q_1,q_2,q_3)$ can be worked out with help of eq. (E.4) in appendix E. We display the value of $\mathcal{L}_6(p_1,p_2,p_3,q_1,q_2,q_3)$ in this low momenta limit in the following equation:

$$
\begin{aligned}
&\mathfrak{L}_6(p_1,p_2,p_3,q_1,q_2,q_3) \approx \int \frac{d\rho}{\rho^5}\, d_0^{(n_f=3)}(\rho)\left(\frac{-i3^3}{64}\right)\left(\frac{4\pi^2\rho^3}{3}\right)^3\Bigg[\\
&\quad d_{abc}\Bigg[-\frac{1}{15}(\overline{\psi}_{1\,R}(p_1)\lambda^a\psi_{1\,L}(q_1))(\overline{\psi}_{2\,R}(p_2)\lambda^b\psi_{2\,L}(q_2))(\overline{\psi}_{3\,R}(p_3)\lambda^c\psi_{3\,L}(q_3))\\
&\quad +\frac{1}{5}((\overline{\psi}_{1\,R}(p_1)\gamma_{\mu\nu}\lambda^a\psi_{1\,L}(q_1))(\overline{\psi}_{2\,R}(p_2)\gamma_{\mu\nu}\lambda^b\psi_{2\,L}(q_2))(\overline{\psi}_{3\,R}(p_3)\lambda^c\psi_{3\,L}(q_3))+(1\leftrightarrow 3)+(2\leftrightarrow 3))\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2}{3}f_{abc}(\overline{\psi}_{1\,R}(p_1)\gamma_{\alpha\beta}\lambda^a\psi_{1\,L}(q_1))(\overline{\psi}_{2\,R}(p_2)\gamma_{\beta\delta}\lambda^b\psi_{2\,L}(q_2))(\overline{\psi}_{3\,R}(p_3)\gamma_{\delta\alpha}\lambda^c\psi_{3\,L}(q_3))\\
&+h\Big\{d_{abc}\Big[-\frac{1}{15}(\overline{\psi}_{1\,R}(p_1)[\overline{\mathcal{O}}(p_1)-\mathcal{O}(q_1)]\lambda^a\psi_{1\,L}(q_1))(\overline{\psi}_{2\,R}(p_2)\lambda^b\psi_{2\,L}(q_2))(\overline{\psi}_{3\,R}(p_3)\lambda^c\psi_{3\,L}(q_3))\\
&+\frac{1}{5}((\overline{\psi}_{1\,R}(p_1)[\overline{\mathcal{O}}(p_1)\gamma_{\mu\nu}-\gamma_{\mu\nu}\mathcal{O}(q_1)]\lambda^a\psi_{1\,L}(q_1))(\overline{\psi}_{2\,R}(p_2)\gamma_{\mu\nu}\lambda^b\psi_{2\,L}(q_2))(\overline{\psi}_{3\,R}(p_3)\lambda^c\psi_{3\,L}(q_3))\\
&+(2\leftrightarrow 3)\\
&+(\overline{\psi}_{1\,R}(p_1)[\overline{\mathcal{O}}(p_1)-\mathcal{O}(q_1)]\lambda^a\psi_{1\,L}(q_1))(\overline{\psi}_{2\,R}(p_2)\gamma_{\mu\nu}\lambda^b\psi_{2\,L}(q_2))(\overline{\psi}_{3\,R}(p_3)\gamma_{\mu\nu}\lambda^c\psi_{3\,L}(q_3)))\Big]-\frac{2}{3}\\
&f_{abc}(\overline{\psi}_{1\,R}(p_1)[\overline{\mathcal{O}}(p_1)\gamma_{\alpha\beta}-\gamma_{\alpha\beta}\mathcal{O}(q_1)]\lambda^a\psi_{1\,L}(q_1))(\overline{\psi}_{2\,R}(p_2)\gamma_{\beta\delta}\lambda^b\psi_{2\,L}(q_2))(\overline{\psi}_{3\,R}(p_3)\gamma_{\delta\alpha}\lambda^c\psi_{3\,L}(q_3))\\
&+(1\leftrightarrow 2)+(1\leftrightarrow 3)\Big\}\Big].
\end{aligned}
$$

$\overline{\mathcal{O}}(p)$ and $\mathcal{O}(p)$ are defined in eq. (4.10).

5. Summary and outlook

In the main body of this paper, we have obtained the following results for noncommutative SU(3) gauge theories with one, two and three light Dirac fermionic flavours, when (Euclidean) time is commutative:

i. There are no solutions at any order in the formal power expansion in $h\theta^{\mu\nu}$ to the noncommutative (anti-)self-duality equations. This result holds for SU(N) as well.

ii. At first order in $h\theta^{\mu\nu}$, ordinary instantons can be given a $\theta^{\mu\nu}$-dependent piece so that the resulting field configuration satisfies the noncommutative Yang-Mills equations. This field configuration — that we have called first-order-in-θ-deformed instanton — has instanton number equal to one, and interpolates, along Euclidean time, between vacua that differ in one unit of the winding number. We have also computed the most general first-order-in-θ-deformed instanton.

iii. We have shown that in the first-order-in-θ-deformed instanton transition a coupling between light left handed and right handed fermions is produced, thus showing that the classical $\mathrm{U}(1)_A$ symmetry of the massless theory is broken at the quantum level. We have computed the 't Hooft vertices that describe this coupling and seen that they receive contributions that are linear in $h\theta^{\mu\nu}$, these contributions being nonlocal even in the low momenta limit.

In the appendices — see appendices A, B, and C, respectively —, we have further obtained that

i. the self-duality equations for noncommutative SU(N) Yang-Mills theory have solutions that are formal power series in $h\theta^{\mu\nu}$ if, and only if, $\theta^{\mu\nu}$ is self-dual. These solutions are the ordinary instantons and multi-instantons. Analogously, the noncommutative anti-self-duality equations for SU(N) have solutions that are formal power series in $h\theta^{\mu\nu}$ if, and only if, $\theta^{\mu\nu}$ is anti-self-dual. The solutions in question are the ordinary anti-instantons and anti-multi-instantons,

ii. that all the noncommutative classical vacua of the noncommutative SU(N) Yang-Mills theory that are formal power series in $h\theta^{\mu\nu}$ are given by the Seiberg-Witten transform of some ordinary vacua $ig(\vec{x})\partial_i g(\vec{x})^\dagger$, where $g(\vec{x})$ is an ordinary gauge transformation, and

iii. that the corrections to the zero mode of the θ-deformed Dirac operator can be worked out explicitly — this we have done — at first order in $h\theta^{\mu\nu}$ for an arbitrary first-order-in-θ-deformed instanton.

The analysis and computations carried out in this paper should be extended at least in two directions. On the one hand, it would be very interesting to see whether, for commutative time, there are topologically nontrivial solutions to the noncommutative Euclidean classical equations of motion that are not formal power series in $h\theta^{\mu\nu}$. This is a highly nontrivial issue since the action of the theory has been defined so far as a formal power series in $h\theta^{\mu\nu}$. Some kind of re-summation of the power series expansion would thus be needed, or, perhaps one should define the Seiberg-Witten map by expanding it in terms of a different object [20]. On the other hand, it will be interesting to work out the second order in $h\theta^{\mu\nu}$ corrections to the instanton density and 't Hooft vertices that we have obtained. This is quite an involved computation since it will demand the use of the constrained instanton method [48, 49] or the valley method [50 – 52] to carry it out. Indeed, as we show in appendix D, there are no topologically nontrivial field configurations that are formal power series in $h\theta^{\mu\nu}$ and leave the noncommutative SU(N) Yang-Mills action stationary at second order in $h\theta^{\mu\nu}$. Actually, at second order in $h\theta^{\mu\nu}$, the size, ρ, of the first-order-in-θ-deformed instanton does not yield a zero mode of the quantum bosonic kinetic term in a background that differs from our first-order-in-θ-deformed instanton by a term quadratic in $h\theta^{\mu\nu}$. Indeed, as can be shown by substituting the r.h.s. of eq. (3.14) in the r.h.s. of eq. (2.10), the noncommutative action acquires a dependence on ρ of the type ρ^{-4}:

$$S_{\rm NCYM} = \frac{8\pi^2}{g^2} + \frac{8h^2\,\pi^2}{7g^2\rho^4}(\theta^{\mu\nu} - \tilde{\theta}^{\mu\nu})^2 + O(h^3\theta^3).$$

Notice that if we add to our first-order-in-θ-deformed instanton in eq. (3.14) an arbitrary piece that is quadratic in $h\theta^{\mu\nu}$, the previous value of $S_{\rm NCYM}$ gets no correction at second order in $h\theta^{\mu\nu}$. Now, ρ gives rise to a quasi-zero mode in the sense of ref. [50], so that the technique developed in the latter reference can be used to compute higher order corrections in $h\theta^{\mu\nu}$.

Acknowledgments

This work has been financially supported in part by MEC through grant FIS2005-02309. The work of C. Tamarit has also received financial support from MEC trough FPU grant AP2003-4034. We would like to express our gratitude to Professors D. Diakonov and D.K. Nielsen for instructing us on some instanton issues.

A. Perturbative solutions to the (anti-)self-duality equations

In this appendix we shall consider an arbitrary $\theta^{\mu\nu}$ and seek for topologically non-trivial solutions, a_μ, to the noncommutative (anti-)self-duality equation — eq. (2.9) — that are given by the formal power series in $h\theta^{\mu\nu}$ in eq. (2.11). Throughout this appendix the noncommutative gauge field and field strength will be defined in terms of the ordinary fields by means of the standard Seiberg-Witten map — see eqs. (2.1) and (2.3). We shall show that for non-vanishing instanton number the solutions we seek for exist if, and only if, $\theta^{\mu\nu}$ is (anti-)self-dual and, further, that these solutions are the ordinary (anti-)instantons and (anti-)multi-instantons. In this appendix $\tilde{v}_{\mu\nu}$ shall denote the dual of a given tensor $v_{\mu\nu}$: $\tilde{v}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}v_{\rho\sigma}$.

The expansion of $a_\mu[h]$ in eq. (2.11) leads to the following expansion of its field strength:

$$f_{\mu\nu}[h] = f^{(0)}_{\mu\nu} + \sum_{l=1}^{\infty} h^l f^{(l)}_{\mu\nu}, \tag{A.1}$$

where $f^{(l)}_{\mu\nu}$ is a homogeneous polynomial in $\theta^{\mu\nu}$ of degree l. Substituting in eq. (2.3), both the previous result and eq. (2.11), one obtains the following expansion of $F_{\mu\nu}[a_\sigma[h]]$ in powers of $h\theta^{\mu\nu}$:

$$F_{\mu\nu}[a_\sigma[h]] = f^{(0)}_{\mu\nu} + \sum_{l=1}^{\infty} h^l f^{(l)}_{\mu\nu} + \sum_{l=1,k=0}^{\infty} h^{l+k} F^{(l,k)}_{\mu\nu}, \tag{A.2}$$

where $F^{(l,k)}_{\mu\nu}$ is given by

$$F^{(l,k)}_{\mu\nu} = \frac{1}{l!k!}\frac{d^k}{dt^k}\frac{d^{l-1}}{d^{l-1}h}\overset{\circ}{F}_{\mu\nu}[a_\sigma[t]]\Big|_{h=t=0}. \tag{A.3}$$

$a_\sigma[t]$ is obtained from $a_\sigma[h]$ in eq. (2.11) by replacing h with t. $\overset{\circ}{F}_{\mu\nu}[a_\sigma]$ is equal to $\frac{d}{dh}F_{\mu\nu}[a_\sigma]$ as defined in eq. (2.4).

We shall show below that $F_{\mu\nu}[a_\sigma[h]]$ as defined above — see the previous equations — is self-dual with non-vanishing Pontrjagin number if, and only if, both $\theta_{\mu\nu}$ and $f_{\mu\nu}[h]$ in eq. (A.1) are self-dual. We shall carry out this proof by induction. At order h^0 and h, the self-duality equation for $F_{\mu\nu}[a_\sigma[h]]$ in eq. (A.2) is equivalent to

$$f^{(0)}_{\mu\nu} = \tilde{f}^{(0)}_{\mu\nu} \tag{A.4}$$

and

$$f^{(1)}_{\mu\nu} + F^{(1,0)}_{\mu\nu} = \tilde{f}^{(1)}_{\mu\nu} + \tilde{F}^{(1,0)}_{\mu\nu}, \tag{A.5}$$

respectively. Since we shall look for solutions with a non-zero Pontrjagin number, we must demand that $f^{(0)}_{\mu\nu}$ does not vanish — see eq. (2.8) — and recall that $f^{(1)}_{\mu\nu}$ does not depend on h. Now, working out the trace over the SU(N) generators on both sides of eq. (A.5) and using eq. (A.4), one obtains the following equation:

$$\sum_a \frac{1}{2}\big[(f^{(0)\,a}_{12})^2 + (f^{(0)\,a}_{13})^2 + (f^{(0)\,a}_{23})^2\big](\theta - \tilde{\theta})_{\mu\nu} = 0, \tag{A.6}$$

where "a" is the colour index of the field strength. Now, for a non-zero $f^{(0)}_{\mu\nu}$, this equation holds if, and only if, $\theta_{\mu\nu}$ is self-dual. For a self-dual $\theta_{\mu\nu}$ and a self-dual $f^{(0)}_{\mu\nu}$, eq. (A.5) boils down to

$$f^{(1)}_{\mu\nu} = \tilde{f}^{(1)}_{\mu\nu}. \tag{A.7}$$

To show that if $\theta_{\mu\nu}$ is self-dual, the self-duality equation for $F_{\mu\nu}[a_\sigma[h]]$ in eq. (A.2) is equivalent to the self-duality equation for $f_{\mu\nu}$ in eq. (A.1), we shall prove first — by induction — the following statement: if $\theta_{\mu\nu}$, $f^{(0)}_{\mu\nu},\ldots,f^{(k-1)}_{\mu\nu}$ are self-dual, then, so are $F^{(m,0)},\ldots,F^{(m,k-1)}$ for all $m \geq 1$.

From eqs. (A.3) and (2.4), obtains that

$$F^{(1,l)}_{\mu\nu} = \sum_{m+n=l} \left(\frac{1}{2}\,\theta^{\kappa\lambda}\{f^{(m)}_{\mu\kappa}, f^{(n)}_{\nu\lambda}\} - \frac{1}{4}\,\theta^{\kappa\lambda}\{a^{(m)}_{\kappa}, [(\partial_\lambda + D_\lambda)f_{\mu\nu}]^{(n)}\}\right).$$

For $l \leq k-1$, the previous expression involves $f^{(m)}_{\mu\nu}$, $m \leq k-1$, which are self-dual by hypothesis; this clearly makes the second term in the previous expression self-dual in μ,ν. The first term is also self-dual in μ,ν as a consequence of the self-duality of $\theta_{\mu\nu}$, the self-duality of $f^{(m)}_{\mu\nu}$, $m \leq k-1$, and the property

$$\epsilon_{\mu\nu\alpha\beta}\epsilon_{\gamma\delta\lambda\beta} = \delta_{\mu\gamma}\delta_{\nu\delta}\delta_{\alpha\lambda} - (\lambda \leftrightarrow \delta) + \delta_{\mu\delta}\delta_{\nu\lambda}\delta_{\alpha\gamma} - (\lambda \leftrightarrow \gamma) + \delta_{\mu\lambda}\delta_{\nu\gamma}\delta_{\alpha\delta} - (\gamma \leftrightarrow \delta). \tag{A.8}$$

This proves the previous statement for $m = 1$. We shall assume in the sequel that the statement holds for $m \leq n-1$. Now, eqs. (A.3) and (2.4) lead to

$$F^{(n,j)}_{\mu\nu} = \\ \frac{1}{n!j!}\frac{d^{n-1}}{dh^{n-1}}\frac{d^j}{dt^j}\left(\tfrac{1}{2}\theta^{\kappa\lambda}\{F_{\mu\kappa}[a_\sigma[t]], F_{\nu\lambda}[a_\sigma[t]]\}_\star - \tfrac{1}{4}\theta^{\kappa\lambda}\{A_\kappa[a_\sigma[t]], (\partial_l + D_l)F_{\mu\nu}[a_\sigma[t]]\}_\star\right)\Big|_{h=t=0}. \tag{A.9}$$

Taking into account the degrees of the derivatives in the previous expression, one readily realizes that the $F^{(i,l)}_{\mu\nu}$'s that occur on the r.h.s. of this equation have $i \leq n-1, l \leq j$. These $F^{(i,j)}_{\mu\nu}$'s are, by hypothesis, self-dual if $j \leq k-1$. One thus concludes that the second term on the r.h.s of eq. (A.9) is self-dual in μ,ν. Eq. (A.8) and the fact that $\theta_{\mu\nu}$ and $F^{(i,l)}_{\mu\nu}$, $i < n$, $l < k$ are self-dual imply that the first term on the r.h.s. of eq. (A.9) is also self-dual. This concludes the proof of the statement made at the beginning of the paragraph starting right below eq. (A.7).

We are now ready to go on with the proof that the only topologically non-trivial solutions to eq. (2.9) that are formal power series in $h\theta^{\mu\nu}$ are the ordinary instantons and multi-instantons. Using eq. (A.2), one readily concludes that the contribution of order h^k to the self-duality equation for $F_{\mu\nu}[a_\sigma[h]]$ reads

$$f^{(k)}_{\mu\nu} + \sum_{m=1}^{k} F^{(m,k-m)}_{\mu\nu} = \tilde{f}^{(k)}_{\mu\nu} + \sum_{m=1}^{k} \tilde{F}^{(m,k-m)}_{\mu\nu}. \tag{A.10}$$

Let us assume that $f^{(0)}_{\mu\nu},\ldots,f^{(k-1)}_{\mu\nu}$ are self-dual, then, the statement made below eq. (A.7) leads to the following result: $F^{(m,k-m)}, 1 \leq m \leq k$ are self-dual. This result and eq. (A.10) imply that whatever the value of $k > 1$

$$f^{(k)}_{\mu\nu} = \tilde{f}^{(k)}_{\mu\nu}.$$

We have thus shown — recall eqs. (A.4) and (A.7) — that $F_{\mu\nu}[a_\sigma[h]]$ in eq. (A.2) is self-dual if, and only if, both $\theta_{\mu\nu}$ and $f^{(k)}_{\mu\nu}$, $\forall k$, are self-dual. Hence, the noncommutative self-duality equation boils down to the ordinary self-duality equation, when one looks for solutions that are formal power series in $h\theta^{\mu\nu}$ and have a non-vanishing Pontrjagin number.

One may readily adapt the procedure carried out above to analyze the existence of solutions to the noncommutative anti-self-duality equation that are power series in $h\theta^{\mu\nu}$. The noncommutative anti-self-duality equation reads $F_{\mu\nu}[a_\sigma[h]] = -\tilde{F}_{\mu\nu}[a_\sigma[h]]$, with $F_{\mu\nu}[a_\sigma[h]]$ defined as in eq. (A.1). It turns out that in the case at hand eq. (A.6) is replaced with

$$\sum_a \frac{1}{2}\big[(f_{12}^{(0)\,a})^2 + (f_{13}^{(0)\,a})^2 + (f_{23}^{(0)\,a})^2\big](\theta+\tilde{\theta})_{\mu\nu} = 0.$$

Hence, only when $\theta_{\mu\nu}$ is anti-self-dual does the previous equation hold for $f^{(0)}_{\mu\nu} \neq 0$. By replacing self-dual objects with anti-self-dual tensors in the analysis above, one finally reaches, with regard to the noncommutative anti-self-duality equation , the conclusion stated at the beginning of this appendix.

B. Classical vacua of noncommutative SU(N) Yang-Mills theory

In this appendix we shall show that, in the temporal gauge in Minkowski space-time — $a_0(t,\vec{x}) = 0$ — and for commutative time — $\theta^{0i} = 0$ —, any gauge field $a_i[h](t,\vec{x})$ that is a formal power series in $h\theta^{\mu\nu}$ defines a classical vacuum of the noncommutative SU(N) Yang-Mills theory if, and only if, there exists $g(\vec{x})\epsilon$ SU(N) such that $a_i[h](t,\vec{x}) = ig(\vec{x})\partial_i g(\vec{x})^\dagger$.

Let us work out first the Hamiltonian of the theory defined by the action in eq. (2.5) rotated to Minkowski space-time and the standard Seiberg-Witten map in eq. (2.1). It can be shown that if $\theta^{0i} = 0$, and $a_0(t,\vec{x}) = 0$, no time derivative occurs in A_i and that A_0 is linear in $\partial_0 a_i \equiv \dot{a}_i$:

$$A_i = a_i + M_i[a_k, \partial_k a_j]; \qquad A_0 = L_0^l[a_i, \partial_i a_j, \partial_k]\dot{a}_l.$$

The field strength $F_{\mu\nu}$ of this noncommutative gauge field reads:

$$\begin{aligned} F_{ij} &= f_{ij} + R_{ij}[a_k, \partial_k a_m], & R_{ij} &= \sum_{l>0} h^l R^{(l)}_{ij}, \\ F_{0i} &= \dot{a}_i + S^c_{ij}[a_k, \partial_i a_j, \partial_k]\dot{a}^c_j, & S^c_{ij} &= \sum_{l>0} h^l S^{c\,(l)}_{ij}. \end{aligned} \tag{B.1}$$

Substituting this expression in eq. (2.5) rotated to Minkowski space-time, one obtains the Lagrangian $\mathcal{L}[t,\vec{y}]$ of the theory in the temporal gauge. This Lagrangian is quadratic in $\dot{a}_i$ so that the conjugate momenta of the field variable a_i are defined as usual:

$$\pi^a_i(t,\vec{x}) = \frac{\delta}{\delta \dot{a}^a_i}\int d^4y\, \mathcal{L}[t,\vec{y}] = -\frac{2}{g^2}\,\mathrm{Tr}\Big\{F^{0i}(t,\vec{x})T^a + \int d^4y\, S_{ij}{}^a[a(y), \partial^y a, \partial^y]\delta^3(\vec{x}-\vec{y})F^{0j}(t,\vec{y})\Big\},$$

$$\int d^4x\, \pi^a_i(t,\vec{x})\dot{a}^a_i(t,\vec{x}) = -\frac{2}{g^2}\,\mathrm{Tr}\int d^4y\, F_{0j}F^{0j}(t,\vec{y}).$$

We then define the hamiltonian, $\mathcal{H}$, of the theory as follows

$$\mathcal{H} = \int d^4x\, \pi_i^a(t,\vec{x})\dot{a}_i^a(t,\vec{x}) - \int d^4x\, \mathcal{L}(t,\vec{x}) = \frac{1}{g^2}\int d^4x\ \mathrm{Tr}\left(F_{0i}F_{0i} + \sum_{i<j} F_{ij}F_{ij}\right). \tag{B.2}$$

Notice that $\mathcal{H}$ is gauge invariant — recall that $\theta^{0i} = 0$ — and that it is equal to $\int d^3\vec{x}\ \mathrm{Tr}\, T^{00}(t,\vec{x})$, where $T^{00}(t,\vec{x})$ is 00 component of the gauge covariant energy-momentum tensor — see [42] for further discussion —: $T^{\mu\nu} = -\frac{1}{g^2}\,\mathrm{Tr}\left(F^{\mu\alpha}\star F^{\nu}{}_{\alpha} + F^{\nu\alpha}\star F^{\mu}{}_{\alpha} - \frac{1}{2}\eta^{\mu\nu}F^{\alpha\beta}\star F_{\alpha\beta}\right)$.

The classical vacua of the theory are defined by those $a_i[t,\vec{x}]$ that minimize the hamiltonian given in eq. (B.2). Since the fields $F_{\mu\nu}$ are self-adjoint, $\mathcal{H}$ is positive-definite. Hence, the vacuum configurations are those which verify

$$F_{0i}(t,\vec{x}) = 0, \qquad F_{ij}(t,\vec{x}) = 0.$$

Let us assume that a_i and $f_{\mu\nu}$ are given by the expansions in non-negative powers of $h\theta^{\mu\nu}$ in eqs. (2.11) and (A.1). Then, eq. (B.1) leads the following expansions:

$$\begin{aligned}
F_{0i} &= \dot{a}_i^{(0)} + \sum_{l>0} h^l \dot{a}_i^{(l)} + \sum_{l>0}\sum_{s\geq 0}\sum_{t\geq 0} h^{l+s+t} S_{ij}^{c\,(l,s)}\dot{a}_i^{c\,(t)} \\
&= \dot{a}_i^{(0)} + \sum_{l>0} h^l\left(\dot{a}_i^{(l)} + \sum_{s=1}^{l}\sum_{t=0}^{l-s} S_{ij}^{c\,(s,t)}\dot{a}_i^{c\,(l-s-t)}\right), \\
S_{ij}^{c\,(l,k)} &= \frac{1}{k!}\frac{d^k}{dh^k} S_{ij}^{c\,(l)}[a^{(0)} + h^l a^{(l)}]|_{h=0}.
\end{aligned}$$

Hence, $F_{0i} = 0$ is equivalent to the following set of equalities:

$$\begin{aligned}
\dot{a}_i^{(0)} &= 0, \\
\dot{a}_i^{(l)} + \sum_{s=1}^{l}\sum_{t=0}^{l-s} S_{ij}^{c\,(s,t)}\dot{a}_i^{c\,(l-s-t)} &= 0, \qquad l\geq 1.
\end{aligned}$$

It is easy to show by induction that the solution to the previous collection of equations reads:

$$\dot{a}_i^{(l)} = 0 \;\Rightarrow\; a_i^{(l)} = a_i^{(l)}(\vec{x}) \;\; l\geq 0.$$

This leads to the conclusion that, in the temporal gauge, the classical vacua of our non-commutative theory are given by time independent gauge fields, say, $a_i[h](\vec{x})$, at least if they can be formally expanded as in eq. (2.11).

Now, for a field configuration of the form of eq. (2.11) with field strength as in eq. (A.1), F_{ij} in (B.1) takes the form:

$$\begin{aligned}
F_{ij} &= f_{ij}^{(0)} + \sum_{l>0} h^l f_{ij}^{(l)} + \sum_{l>0,k\geq 0} h^{l+k} F_{ij}^{(l,k)} = f_{ij}^{(0)} + \sum_{l>0} h^l\left(f_{ij}^{(l)} + \sum_{k=1}^{l} F_{ij}^{(k,l-k)}\right), \\
F_{ij}^{(l,k)} &= \frac{1}{k!}\frac{d^k}{dh^k}F_{ij}^{(l)}[a_i^{(0)} + h^l a_i^{(l)}]|_{h=0}, \quad F_{ij}^{(l)}[a_i] = \frac{1}{l!}\frac{d^{l-1}}{dh^{l-1}}\left[\frac{dF_{ij}}{dh}\right][a_i^{(0)} + t^l a_i^{(l)}]|_{h=0,t\to h}
\end{aligned}$$

where, $\frac{dF_{\mu\nu}}{dh}$ is given in eq. (2.4). We thus conclude that $F_{ij}=0$, is equivalent to the set of equations

$$\begin{aligned} &f^{(0)}_{\mu\nu}=0,\\ &f^{(l)}_{ij}+\textstyle\sum_{k=1}^{l}F^{(k,l-k)}_{ij}=0,\quad l\geq 1. \end{aligned} \tag{B.3}$$

Now, it can be shown by induction that $f_{\mu\nu}=0$ implies $F^{(l)}_{\mu\nu}[a_\rho]=0$, $\forall l$. Using this result, one can prove that

$$F^{(l,k)}_{ij}=0,\quad \text{if}\quad f^{(n)}_{ij}=0,\quad 0\leq n\leq k.$$

Furnished with this result, one readily shows that the solution, $a_i(\vec{x})$, to eq. (B.3) must satisfy

$$f^{(l)}_{ij}=0,\quad l\geq 0\quad \Leftrightarrow\quad f_{ij}=0,$$

i.e., $a_i(\vec{x})$ is a pure gauge: $a_i(\vec{x})=ig(\vec{x})\partial_i g^\dagger(\vec{x})$.

C. Zero modes in the instanton background

In this appendix, we shall work out the zero modes — that we shall call bosonic zero modes — of the operator $\mathcal{M}^{ab}_{\mu\nu}[a^{(\mathrm{gsing})}_\sigma]$ in eq. (4.4) and the zero mode — referred to as the fermionic zero mode — of the operator $\mathcal{K}[a^{(\mathrm{gsing})}_\mu]$ defined in eq. (4.2).

C.1 Bosonic zero modes

As in the ordinary case — see ref. [46] —, the zero modes, $\delta^i a_\mu(x)$, of the operator $\mathcal{M}^{ab}_{\mu\nu}[a^{(\mathrm{gsing})}_\sigma]$ in eq. (4.4) are given by

$$\delta^i a_\mu=\frac{\partial a^{(\mathrm{gsing})}_\mu(x,\gamma_j)}{\partial\gamma_i}-\mathfrak{D}[a^{(\mathrm{gsing})}_\sigma]_\mu\Omega^i.$$

$\{\gamma_i\}$ denote the collective coordinates of $a^{(\mathrm{gsing})}_\mu$ and Ω^i is a gauge transformation that makes $\delta^i a_\mu$ satisfy the background field gauge condition $\mathfrak{D}[a^{(\mathrm{gsing})}_\sigma]_\mu\delta^i a_\mu=0$.

There are, of course, twelve zero modes — as many as the number of dimensions of the moduli space of our first-order-in-θ-deformed instanton. Let U denote the rigid SU(3) transformation that relates $a^{(\mathrm{gsing})}_\nu$ and $a^{(\mathrm{sing})}_\nu$ — see eq. (3.15) — and let a^{BPST}_ν denote the upper-left-hand corner embedding in SU(3) of the ordinary BPST instanton in the singular gauge. Then, the zero modes we look for read

$$\begin{aligned}
\delta^\mu a_\nu&=\frac{\partial a^{(\mathrm{gsing})}_\nu}{\partial x_{0\mu}}+U\big(\mathfrak{D}[a^{\mathrm{BPST}}_\sigma]_\nu a^{\mathrm{BPST}}_\mu\big)U^\dagger,\\
\delta^\rho a_\nu&=\frac{\partial a^{(\mathrm{gsing})}_\nu}{\partial\rho},\\
\delta^a a_\mu&=U\left(\mathfrak{D}[a^{\mathrm{BPST}}_\sigma]_\mu\left[\frac{2r^2}{r^2+\rho^2}T^a\right]\right)U^\dagger,\quad a\in\{1,2,3\},\\
\delta^\alpha a_\mu&=U\left(\mathfrak{D}[a^{\mathrm{sing}}_\sigma]_\mu\left[2\sqrt{\frac{r^2}{r^2+\rho^2}}\,T^\alpha+\frac{ih}{4\sqrt{3}\rho^2}\frac{(4\rho^2r^3+3r^5)}{(r^2+\rho^2)^{5/2}}(\theta-\tilde{\theta})_{\rho\sigma}\bar{\eta}_{a\rho\sigma}f^{a\alpha c}f^{8cd}T^d\right]\right)U^\dagger
\end{aligned}$$

$$\alpha \in \{4,5,6,7\}, \quad a,c,d \in \{1,2,3\}.$$

We have used the following notation $r \equiv \sqrt{(x-x_0)^2}$. T^a, $a=1,2,3$ are given by the embedding the of generators of SU(2) into the upper-left-hand corner of he generators of SU(3) in the fundamental representation. The generators T^4 and T^5, and T^6 and T^7, form doublets under the action of the SU(2) subgroup generated by T^a, $a=1,2,3$. The remaining generator T^8 is a singlet under the action of this SU(2) subgroup. The limit $h\theta^{\mu\nu} \to 0$ of the zero modes above yields the ordinary zero modes computed in ref. [46].

The Jacobian, $J(\gamma)$, of the collective coordinates is given by the following expression

$$J(\gamma) = \prod_i \frac{1}{\sqrt{2\pi}}\sqrt{\det \mathfrak{g}(\gamma_k)},$$

where the metric, $\mathfrak{g}(\gamma_k)$, of the moduli space is given by:

$$\mathfrak{g}^{ij}(\gamma_k) = \frac{2}{g^2}\,\mathrm{Tr}\int d^4x\,\delta^i a_\nu(x,\gamma_k)\delta^j a_\nu(x,\gamma_k).$$

Of course, there are no contributions to $\mathfrak{g}(\gamma_k)$ nor $J(\gamma)$ that are linear in $h\theta^{\mu\nu}$: they are proportional to $h\theta_{\mu\nu}\,g_{\mu\nu}$, $g_{\mu\nu}$ being the space-time metric. Hence,

$$J(\gamma) = \frac{1}{(2\pi)^6}\sqrt{\det g} = \frac{2^{14}\pi^6\rho^3}{g^{12}}\big[128\,\rho^4 - 25\,h^2\,(\theta_{\mu\nu}-\tilde{\theta}_{\mu\nu})^2\big] = \frac{2^{14}\pi^6\rho^7}{g^{12}} + O(h^2\theta^2).$$

C.2 Fermionic zero mode

It was shown in ref. [41] that, as a consequence of there being an $\mathrm{U}(1)_A$ anomaly, the index of the operator $\mathcal{K}[a_\mu]$ is one if a_μ has Pontrjagin number equal to one. Hence, a least in perturbation theory of $h\theta^{\mu\nu}$, $\mathcal{K}[a_\mu]$ has a unique zero mode — which turns out to be right handed — with unit norm. In this appendix we shall explicitly construct such zero mode at first order in $h\theta^{\mu\nu}$ when $a_\mu = a_\mu^{(\mathrm{gsing})}$. The unit norm zero mode of $\mathcal{K}[a_\mu^{(\mathrm{gsing})}]$ defined in eq. (4.2) can be obtained from the unit norm zero mode of $\mathcal{K}[a_\mu^{(\mathrm{sing})}]$ by applying an appropriate rigid SU(3) transformation. Let us then solve $\mathcal{K}[a_\mu^{(\mathrm{sing})}]\psi_0 = 0$ at fir st order in $h\theta^{\mu\nu}$. To do so, we shall expand ψ_0 in positive powers of $h\theta^{\mu\nu}$ with coefficients that are square integrable functions. Up to first order in $h\theta^{\mu\nu}$, we have

$$\psi_0 = \psi_0^{(0)} + h\psi_0^{(1)} + O(h^2\theta^2),$$

where $\psi_0^{(1)}$ is linear in $\theta^{\mu\nu}$, and $\psi_0^{(0)}$ and $\psi_0^{(1)}$ satisfy the following equations:

$$\begin{aligned}
\not{D}[a_\sigma^{\mathrm{BPST}}]\psi_0^{(0)} &= 0, \\
\not{D}[a_\sigma^{\mathrm{BPST}}]\psi_0^{(1)} &= i\,\gamma_\mu b_\mu \psi_0^{(0)} + \frac{1}{2}\theta_{\alpha\beta}\gamma_\mu f_{\mu\alpha}^{\mathrm{BPST}} D[a_\sigma^{\mathrm{BPST}}]_\beta \psi_0^{(0)} - \frac{1}{8}\theta_{\alpha\beta}\gamma_\mu(\mathfrak{D}[a_\sigma^{\mathrm{BPST}}]_\mu f_{\alpha\beta}^{\mathrm{BPST}})\psi_0^{(0)}.
\end{aligned} \tag{C.1}$$

$f_{\alpha\beta}^{\mathrm{BPST}}$ denotes the field strength of a_μ^{BPST}, both being in the singular gauge. Recall that $a_\mu^{(\mathrm{sing})} = a_\mu^{\mathrm{BPST}} + h\,b_\mu$, with

$$a_\mu^{\mathrm{BPST}}(x) = \frac{\overline{\eta}_{a\mu\nu}\,(x-x_0)_\nu\rho^2}{(x-x_0)^2[(x-x_0)^2+\rho^2]}\,\tau^a,$$

$$b_\mu(x) = \frac{2}{\sqrt{3}}(\theta - \tilde{\theta})_{\mu\alpha}(x-x_0)_\alpha \frac{(x-x_0)^2 + 3\rho^2}{((x-x_0)^2+\rho^2)^3} T^8.$$

The first equality in eq. (C.1) is the ordinary zero mode equation. Its solution is well known [38]. It is the following spinor with positive chirality:

$$\psi^{(0)}_{0,im} = \frac{\rho}{\pi r(r^2+\rho^2)^{3/2}}\left[\left(\frac{1+\gamma_5}{2}\right)(\not{x} - \not{x}_0)\right]_{ij} \epsilon^{jm}. \tag{C.2}$$

Note that i, j stand for spinor indices and m, n for colour indices.

Now, using the properties of the Gell-Mann matrices, one may show that the second equality in eq. (C.1) implies that the third colour component of $\psi_0^{(1)}$ must vanish, if it vanishes at infinity. Hence, the second equality in eq. (C.1) can be reduced to a equation with colour indices belonging to SU(2) — in the fundamental representation — upon replacing T^8 in b_μ with $\frac{1}{2\sqrt{3}}\mathbb{I}$. This we shall do.

The term on the far r.h.s of eq. (C.1) can be expressed as follows:

$$\not{D}^{\rm BPST}\left(-\frac{h}{8}\theta_{\alpha\beta} f_{\alpha\beta}\psi_0^{(0)}\right).$$

Let $\psi^{(1b)}$ be defined by the following equations:

$$\psi_0^{(1)} \equiv \psi_0^{(1a)} + \psi_0^{(1b)}; \quad \psi^{(1a)} = -\frac{1}{8}\theta_{\alpha\beta} f_{\alpha\beta}\psi_0^{(0)}. \tag{C.3}$$

In terms of $\psi_0^{(1b)}$, eq. (C.1) reads

$$\not{D}[a_\sigma^{\rm BPST}]\psi_0^{(1b)} = i\,\gamma_\mu b_\mu \psi_0^{(0)} + \frac{h}{2}\theta_{\alpha\beta}\gamma_\mu f^{\rm BPST}_{\mu\alpha} D[a^{\rm BPST}_\sigma]_\beta \psi_0^{(0)} \equiv \mathfrak{R}. \tag{C.4}$$

To solve the previous equation we shall adapt to our case the technique developed in ref. [47]. Let us decompose first $\psi_0^{(0)}$, $\mathfrak{R}$ and $\psi_0^{(1b)}$ into its positive, R, and negative, L, chirality parts:

$$\psi_0^{(0)} \equiv \begin{bmatrix} 0 \\ \psi^{(0)}_{0,R}\end{bmatrix}, \qquad \mathfrak{R} \equiv \begin{bmatrix}\mathfrak{R}_L \\ 0\end{bmatrix}, \qquad \psi_0^{1b} \equiv \begin{bmatrix}\psi_L^{1b} \\ \psi_R^{1b}\end{bmatrix}.$$

Then, when expressed in terms of the bi-spinors $\psi'_{L/R}$ and $\mathfrak{R}'_L$, defined by the equations

$$\begin{aligned}(\psi'_{L/R})_{im} &\equiv (\psi^{1b}_{L/R})_{jm}(\sigma^2)_{ji} \\ (\mathfrak{R}'_L)_{im} &\equiv (\mathfrak{R}_L)_{jm}(\sigma^2)_{ji},\end{aligned} \tag{C.5}$$

eq. (C.4) yields

$$\begin{aligned}(\partial_\mu - i a_\mu^{\rm BPST})\psi'_L\alpha_\mu &= 0 \\ (\partial_\mu - i a_\mu^{\rm BPST})\psi'{}_R\overline{\alpha}_\mu &= \mathfrak{R}'_L.\end{aligned}$$

where α_μ is defined in appendix E.

It is well known [39] that there is no non-vanishing square integrable ψ'_L that is solution to the first equation in eq. (C.6). To find ψ'_R that verifies the second equality in eq. (C.6), let us first express ψ'_R, $\mathfrak{R}'_L$ and $a_\mu^{\rm BPST}$in terms of α_μ, $\overline{\alpha}_\mu$ — see appendix E —, $\sigma_\mu = (\overrightarrow{\sigma}, i)$ and $\sigma_{\mu\nu}$, respectively:

$$(\psi'_R)_{im} = M_\mu(\alpha_\mu)_{mi}; \quad (\mathfrak{R}'_L)_{im} = k_\mu(\overline{\alpha}_\mu)_{mi} = k_4\,\delta_{im} + ik_{\mu\nu}\,(\sigma_{\mu\nu})_{mi}\,/\,k_{\mu\nu} = \frac{1}{2}\overline{\eta}_{b\mu\nu}k_b,$$
$$a_\mu^{\rm BPST} = -\sigma_{\mu\nu}\varphi_\nu,\,; \quad \varphi_\nu = \partial_\nu \log\lambda,\,; \quad \lambda = 1 + \frac{\rho^2}{(x-x_0)^2}. \tag{C.6}$$

By substituting the previous expressions in eq. (C.6) and then using the equalities in eq. (E.1), one obtains the following equation:

$$i\sigma_{\mu\nu}\,(2\partial_\nu M_\mu - \varphi_\nu M_\mu - k_{\mu\nu}) + \left(\partial_\mu M_\mu + \frac{3}{2}\varphi_\nu M_\nu - k_4\right) = 0. \tag{C.7}$$

Taking traces leads to the conclusion that each summand in the previous equation must vanish independently. The following definitions

$$M_\mu \equiv \lambda^{1/2}N_\mu, \quad N_\mu \equiv \partial_\mu\frac{\phi}{\lambda} + \partial_\nu X_{\nu\mu}, \quad X_{\nu\mu}\ \text{anti self-dual}, \tag{C.8}$$

allow us to show that eq. (C.7) is equivalent to the following pair of equalities

$$\begin{aligned}\Box X_{\mu\nu} &= -\frac{2k_{\mu\nu}}{\lambda^{1/2}},\\ \Box\phi &= \frac{k_4}{\lambda^{5/2}} - \frac{\partial_\mu(\lambda^2\partial_\nu X_{\nu\mu})}{\lambda},\end{aligned} \tag{C.9}$$

where

$$\begin{aligned}k_4 &= 0,\\ k_{\rho\sigma} &= m_{\rho\sigma} - \frac{1}{2}\,\epsilon_{\rho\sigma\alpha\beta}m_{\alpha\beta},\\ m_{\rho\sigma} &= -\frac{\sqrt{2}}{2\pi r(r^2{+}\rho^2)^{9/2}}\left[\lambda^3(\theta_{\rho\sigma}r^2 - 4\Delta x_\beta\Delta x_\rho\theta_{\sigma\beta} + 4\Delta x_\beta\Delta x_\sigma\theta_{\rho\beta}) + \frac{1}{6}\lambda\theta_{\rho\sigma}r^2(r^2+3\lambda^2)\right],\\ \Delta x_\sigma &\equiv (x-x_0)_\sigma.\end{aligned}$$

The general solution to eq. (C.9) is the sum of a particular solution and the general solution of the corresponding homogeneous set of equations. A solution to the first equality in question is

$$X_{\mu\nu}(x)_{\rm part} = \frac{1}{2\pi}\int d^4y\;\frac{1}{(x-y)^2}\,\frac{k_{\mu\nu}(y)}{\sqrt{\lambda(y)}}.$$

Recall that $-\frac{1}{4\pi}\frac{1}{(x-y)^2}$ is a Green function of the Laplace operator in four dimensions. By substituting $k_4 = 0$ and the previous value of $X_{\mu\nu}(x)$ in the r.h.s. of the second equation in eq. (C.9), one shows that this r.h.s. vanishes. Hence, $\phi_{\rm part}(x) = 0$ and $X_{\mu\nu}(x)_{\rm part}$ above constitute a particular solution to eq. (C.9).

Adding to this particular solution an appropriate solution to the corresponding homogeneous set of equations is equivalent to adding $c(\theta)\,\psi_0^{(0)}(x)$ to the $\psi_0^{(1b)}$ — call it $\psi_{0\,part}^{(1b)}$ —

constructed from $\phi_{\rm part}(x) = 0$ and $X_{\mu\nu}(x)_{\rm part}$ right above by using eqs. (C.8), (C.6), (C.5) and (C.4). $c(\theta)$ is an arbitrary coefficient linear in $\theta^{\mu\nu}$. Normalization to 1 of ψ_0 renders $c(\theta)$ equal to zero, for $\psi_0^{(0)}$ has unit norm and $\psi^{(1)}_{0\,part} = \psi_0^{(1a)} + \psi^{(1b)}_{0\,part}$ is orthogonal to $\psi_0^{(0)}$ — $(\psi_0^{(0)}, \psi^{(1)}_{0\,part})$ is proportional to $\theta_{\mu\nu}\, g_{\mu\nu}$, $g_{\mu\nu}$ being the space-time metric. We thus conclude that $\psi_0^{(1b)}$ in eq. (C.3) is equal to $\psi^{(1b)}_{0\,part}$, so that we finally have

$$\begin{aligned}\psi^{(1)}_{0,im} &= \psi^{(1a)}_{0,im} + \psi^{(1b)}_{0,im}\\ &= \frac{1}{\pi\,\rho\,r\,(r^2+\rho^2)^{7/2}}\left[\left(\frac{1+\gamma_5}{2}\right)\gamma_\sigma\right]_{ij}\\ &\quad\times\Big\{\frac{i}{12}(x-x_0)_\rho\left[(\theta-\tilde\theta)_{\rho\sigma}(4r^4+14r^2\rho^2) - 6\rho^4(\theta+\tilde\theta)_{\rho\sigma}\right]\epsilon_{jm}\\ &\quad+2\,\rho^4\,\theta_{\alpha\gamma}(x-x_0)_\sigma\left[\frac{x_\alpha x_\nu}{r^2} - \frac{1}{4}\,\delta_{\alpha\nu}\right]\epsilon_{jn}\,\tau_{\gamma\nu,mn}\Big\}.\end{aligned}\tag{C.10}$$

$\tau_{\mu\nu}$ are the analogs of $\sigma_{\mu\nu}$ in SU(2) colour space.

Finally, by acting with the appropriate SU(3) transformation, U, on $\psi_0^{(0)}$ and $\psi_0^{(1)}$ in eqs. (C.2) and (C.10), one obtains the unit norm zero mode of $\mathcal{K}[a_\mu^{(\rm gsing)}]$. This zero mode reads

$$\psi_0 \equiv \psi^0 + \psi^{1a} + \psi^{1b},\tag{C.11}$$

where, writing these right handed spinors in two-component notation, we have:

$$\begin{aligned}\psi^0{}_{im} &= \phi(x)\,(x-x_0)_\mu(\overline{\alpha}_\mu)_{ij}\epsilon_{jn}U_{mn},\\ \psi^{1a}{}_{im} &= h\left[\Gamma_{\alpha\gamma}(x)\,(x-x_0)_\alpha(x-x_0)_\nu + \Lambda_{\nu\gamma}(x)\right]\,(x-x_0)_\mu(\overline{\alpha}_\mu)_{ij}\epsilon_{jo}(\tau_{\gamma\nu})_{no}U_{mn},\\ \psi^{1b}{}_{im} &= h\chi_{\alpha\sigma}(x)\,(x-x_0)_\alpha(\overline{\alpha}_\sigma)_{ij}\epsilon_{jn}U_{mn}.\end{aligned}\tag{C.12}$$

The functions $\phi(x)$, $\Gamma_{\alpha\gamma}(x)$, $\Lambda_{\alpha\gamma}(x)$ and $\chi_{\alpha\sigma}(x)$ are defined thus

$$\begin{aligned}\phi(x) &= \frac{\rho}{\pi r(r^2+\rho^2)^{3/2}},\quad \Gamma_{\alpha\gamma}(x) = \frac{2\theta_{\alpha\gamma}\rho^3}{\pi r^3(r^2+\rho^2)^{7/2}},\quad \Lambda_{\alpha\gamma}(x) = -\frac{\theta_{\alpha\gamma}\rho^3}{2\pi r(r^2+\rho^2)^{7/2}},\\ \chi_{\alpha\sigma}(x) &= \frac{i}{12\pi\rho r(r^2+\rho^2)^{7/2}}\left[(\theta-\tilde\theta)_{\alpha\sigma}(4r^4+14r^2\rho^2) - 6\rho^4(\theta+\tilde\theta)_{\alpha\sigma}\right].\end{aligned}\tag{C.13}$$

Recall that $r = \sqrt{(x-x_0)^2}$.

D. Topologically non-trivial field configurations at higher orders in $\theta^{\mu\nu}$

In this appendix we shall show that, if $\theta^{4i} = 0$, $i = 1, 2, 3$, no topologically nontrivial field configurations can be found that **a)** are formal power series in $h\theta^{\mu\nu}$ and **b)** at second order in $h\theta^{\mu\nu}$ solve the equations of motion of noncommutative SU(3) Yang-Mills theory. To show it, we shall use the technique devised in ref. [53]. Thus, we shall consider the behaviour of action in eq. (2.5) under the following infinitesimal changes of scale:

$$a'_\mu = \lambda a_\mu(\lambda x),\quad \lambda = 1 + \delta\lambda,\tag{D.1}$$

where a_μ satisfies the equations of motion. The action can be written as:

$$S_{\rm NCYM} = \sum_{n=0}^{\infty} h^n S^{(n)} = \sum_{n=0}^{\infty} h^n \int d^4x\, \mathcal{L}^{(n)}[a,\partial], \tag{D.2}$$

where $\mathcal{L}^{(n)}$ is the term of the lagrangian of order h^n, which, due to the fact that $\mathcal{L}^{(n)}$ contains n powers of $\theta^{\mu\nu}$, is a polynomial of mass degree $4+2n$ in a_μ and ∂_μ. Hence,

$$\int d^4x\, \mathcal{L}^{(n)}[\lambda a(\lambda x), \partial^x] = \lambda^{-4} \int d^4y\, \mathcal{L}^{(n)}[\lambda a(y), \lambda\partial^y] = \lambda^{2n} \int d^4y\, \mathcal{L}[a(y), \partial^y].$$

Thus, under the change in eq. (D.1),$S^{(n)}[a]$ in eq. (D.2) changes as follows:

$$S^{(n)\prime}[a'] = \lambda^{2n} S^{(n)}[a].$$

Since we are assuming that the original field configuration a_μ in eq. (D.1) is a solution to the equations of motion, the following equivalent equations hold:

$$S[a] = S[\lambda a(\lambda x)] + O(\delta\lambda^2) \Leftrightarrow \delta\lambda \sum_{n=0}^{\infty} 2n\, h^n S^{(n)}[a] = O(\delta\lambda^2)\ \ \forall\lambda \ \ \Leftrightarrow \ \sum_{n=1}^{\infty} 2n\, h^n\, S^{(n)}[a] = 0. \tag{D.3}$$

Now, let our solution to the equations of motion, a_μ, be given by the following power series: $a_\mu = \sum_{n=0}^{\infty} h^n a_\mu^{(n)}$. By substituting this power series in $S^{(n)}$ in eq. (D.2), one obtains $S^{(n)}[a] = \sum_{k=0}^{\infty} S^{(n,k)}[a]$, $S^{(n,m)} = \frac{1}{m!}\frac{d^m}{dh^m} S^{(n)}[a_\mu]|_{h=0}$. Combining this result with the equality on the far right of eq. (D.3), one ends up with

$$0 = \sum_{k=1}^{\infty} h^k \left(\sum_{m=0}^{k-1} (k-m) S^{(k-m,m)} \right) \quad \Leftrightarrow \quad 0 = \sum_{m=0}^{k-1} (k-m) S^{(k-m,m)} = 0, \forall k \geq 1.$$

For the action, $S_{\rm NCYM}$, to be stationary up to order h^l, the previous identities have to be verified for $k \leq l$. In our case, we want the action to be stationary at order h^2, so that we should check if the following equalities hold:

$$S^{1,0} = 0, \quad 2S^{2,0} + S^{1,1} = 0. \tag{D.4}$$

From eq. (2.5) and eq. (2.8), one concludes that $S_{\rm NCYM}$ for a field configuration with a well defined topological charge n reads:

$$S_{\rm NCYM} = \frac{1}{2g^2}\,{\rm Tr} \int d^4x \left[f_{\mu\nu}\tilde{f}_{\mu\nu} + \frac{1}{2}(F_{\mu\nu} - \tilde{F}_{\mu\nu})^2 \right] = \frac{8\pi^2 n}{g^2} + \frac{1}{4g^2}\,{\rm Tr} \int d^4x\, (F_{\mu\nu} - \tilde{F}_{\mu\nu})^2. \tag{D.5}$$

$F_{\mu\nu}$ is given by the Seiberg-Witten map as power series: $F_{\mu\nu}[a_\rho] = f_{\mu\nu} + \sum_{l>0} h^l F^{(l)}$. When evaluating these terms for the solution $a_\mu = \sum_{n=0}^{\infty} h^n a_\mu^{(n)}$ we get again $F_{\mu\nu}^{(n)}[a_\rho] = \sum_{k=0}^{\infty} F_{\mu\nu}^{(n,k)}[a_\rho]$, $F_{\mu\nu}^{(n,m)}[a_\rho] = \frac{1}{m!}\frac{d^m}{dh^m} F_{\mu\nu}^{(n)}[a_\rho]|_{h=0}$ and $f_{\mu\nu} = f_{\mu\nu}^{(0)} + \sum_{k=1}^{\infty} h^k\, f_{\mu\nu}^{(k)}$. Hence, $S^{(1,0)}$, $S^{(1,1)}$ and $S^{(2,0)}$ in eqs. (D.2) and (D.4) are given by

$$S^{(1,0)} = \frac{1}{2g^2}\,{\rm Tr} \int d^4x\, (f_{\mu\nu}^{(0)} - \tilde{f}_{\mu\nu}^{(0)})(F_{\mu\nu}^{(1,0)} - \tilde{F}_{\mu\nu}^{(1,0)}),$$

$$
\begin{aligned}
S^{(1,1)} &= \frac{1}{2g^2}\,\mathrm{Tr}\int d^4x \left[(f^{(1)}_{\mu\nu}-\tilde{f}^{(1)}_{\mu\nu})(F^{(1,0)}_{\mu\nu}-\tilde{F}^{(1,0)}_{\mu\nu}) + (f^{(0)}_{\mu\nu}-\tilde{f}^{(0)}_{\mu\nu})(F^{(1,1)}_{\mu\nu}-\tilde{F}^{(1,1)}_{\mu\nu})\right],\\
S^{(2,0)} &= \frac{1}{2g^2}\,\mathrm{Tr}\int d^4x\,(f^{(0)}_{\mu\nu}-\tilde{f}^{(0)}_{\mu\nu})(F^{(2,0)}_{\mu\nu}-\tilde{F}^{(2,0)}_{\mu\nu}) + \frac{1}{4g^2}\,\mathrm{Tr}\int d^4x\,(F^{(1,0)}_{\mu\nu}-\tilde{F}^{(1,0)}_{\mu\nu})^2.
\end{aligned}
$$

In section 2, we saw that the most general solution to the equations of motion at order h is given by

$$
a_\mu = U\left(a^{\mathrm{BPST}}_\mu + h b^8_\mu T^8 + h\sum_{a=1}^{7} b^a_\mu T^a\right)U^\dagger, \tag{D.6}
$$

where b^8_μ is given in eq. (3.14), $h\sum_{a=1}^{7} b^a_\mu T^a$ is any linear combination — with coefficients linear in $h\theta^{\mu\nu}$ — of the ordinary bosonic zero modes [46] — i.e., the solutions to eq. (3.11) — and U is a rigid SU(3) transformation. For a field configuration of this type, we have $S^{(1,0)}=0$, for $f^{\mathrm{BPST}}_{\mu\nu}=\tilde{f}^{\mathrm{BPST}}_{\mu\nu}$. The first condition in eq. (D.4) is thus automatically satisfied. Notice that eq. (D.5) tell us that any contribution of order $h^2\theta^2$ to the classical solution in eq. (D.6) yields a contribution of order $h^3\theta^3$ to the action S_{NCYM}. We will show next that the second condition in eq. (D.4) is violated by the solution in eq. (D.6), so that it is impossible to find a field configuration with non-zero topological charge that makes the action stationary up to order $h^2\theta^2$. First, $F^{(1,1)}$ and $F^{(2,0)}$ do not contribute neither to $S^{(1,1)}$ nor $S^{(2,0)}$. Now, taking a closer look at the structure constants of SU(3), one sees that the contribution to $S^{(1,1)}$ of $h\sum_{a=1}^{7} b^a_\mu T^a$ is zero:

$$
S^{(1,1)} = \frac{1}{2g^2}\,\mathrm{Tr}\int d^4x\,(f^{(1),8}_{\mu\nu}T^8 - \tilde{f}^{(1),8}_{\mu\nu}T^8)(F^{(1,0)}_{\mu\nu}-\tilde{F}^{(1,0)}_{\mu\nu}),\;\; f^{(1),8}_{\mu\nu} = hU(\partial_\mu b^8_\nu - \partial_\nu b^8_\mu)U^\dagger.
$$

By evaluating $S^{(1,1)}$ and $S^{(2,0)}$ for the field configuration in eq. (D.6), one obtains

$$
\begin{aligned}
S^{(1,1)}[a_\mu] &= -\frac{8\pi^2}{7g^2\rho^4}(\theta_{\mu\nu}-\tilde{\theta}_{\mu\nu})^2\\
S^{(2,0)}[a_\mu] &= \frac{12\pi^2}{7g^2\rho^4}(\theta_{\mu\nu}-\tilde{\theta}_{\mu\nu})^2
\end{aligned}
\qquad 2S^{(2,0)} + S^{(1,1)} = \frac{16\pi^2}{7g^2\rho^4}(\theta_{\mu\nu}-\tilde{\theta}_{\mu\nu})^2,
$$

so that eq. (D.4) is violated and, furthermore, this happens independently of the arbitrary part of the solution in eq. (D.6). Hence, the only way to make $2S^{(2,0)}+S^{(1,1)}$ zero is by taking ρ to infinity, which would turn our solution into the trivial one.

This conclusion still holds for the most general Seiberg-Witten map at order h, given by eq. (2.15). It turns out that the expression for $S^{(1)}$ obtained with this map is the same as the one derived with the standard map for arbitrary a_μ tending to zero at infinity. Therefore, the field configuration in eq. (D.6) is the most general classical solution at order $h\theta^{\mu\nu}$ with unit topological charge for an arbitrary Seiberg-Witten map. When checking whether the conditions in eq. (D.4) hold, the values of $S^{(1,0)}$ and $S^{(1,1)}$ are unchanged since so is $S^{(1)}$. It can also be seen that, for field configurations that are θ dependent deformations of the ordinary instanton, the value of $S^{(2,0)}$ is the same for all the Seiberg-Witten maps. Therefore the conditions in eq. (D.4) are always violated, and this concludes the proof of the statement made at the beginning of this section.

E. Some conventions and formulae

In this appendix sundry formulae are collected.

E.1 Spinor matrices

$$\sigma_\mu = (\overrightarrow{\sigma}, i), \quad \overline{\sigma}_\mu = (-\overrightarrow{\sigma}, i), \quad \alpha_\mu = (-i\overrightarrow{\sigma}, \mathbb{I}) = -i\sigma_\mu^-, \quad \overline{\alpha}_\mu = (i\overrightarrow{\sigma}, \mathbb{I}) = i\sigma_\mu^+,$$
$$\overline{\sigma}_{\mu\nu} = \frac{1}{4i}(\overline{\alpha}_\mu \alpha_\nu - \overline{\alpha}_\nu \alpha_\mu) = \frac{1}{2}\eta_{a\mu\nu}\sigma_a; \; \sigma_{\mu\nu} = \frac{1}{4i}(\alpha_\mu \overline{\alpha}_\nu - \alpha_\nu \overline{\alpha}_\mu) = \frac{1}{2}\overline{\eta}_{a\mu\nu}\sigma_a,$$
$$\alpha_\mu \overline{\alpha}_\nu = g_{\mu\nu} + 2i\sigma_{\mu\nu}, \tag{E.1}$$

and analogously for the SU(2) generators τ^a, $a = 1, 2, 3$.

$$\epsilon_{12} = +1, \qquad \epsilon_{im}\epsilon_{jn} = \frac{1}{8}(\sigma_\mu^- \sigma_\nu^+)_{ij}(\tau_\mu^- \tau_\nu^+)_{mn}.$$

$$\gamma_\mu = \begin{bmatrix} & \alpha_\mu \\ \overline{\alpha}_\mu & \end{bmatrix}, \quad \gamma_5 = -\gamma_1\gamma_2\gamma_3\gamma_4 = \begin{bmatrix} -\mathbb{I} & \\ & \mathbb{I} \end{bmatrix}, \quad \gamma_{\mu\nu} = \tfrac{1}{4i}[\gamma_\mu, \gamma_\nu]$$

E.2 SU(3) averages

$$\begin{aligned}
\int dU &= 1, \\
\int dU U_{ia} U^\dagger{}_{jb} &= \frac{1}{3}\delta_{ja}\delta_{ib}, \\
\int dU U_{i_1a_1} U^\dagger{}_{j_1b_1} U_{i_2a_2} U^\dagger{}_{j_2b_2} &= \frac{1}{3^2}\delta_{j_1a_1}\delta_{i_1b_1}\delta_{j_2a_2}\delta_{i_2b_2} \\
&\quad + \frac{1}{4\cdot 8}(\lambda^a)_{j_1a_1}(\lambda^a)_{j_2a_2}(\lambda^b)_{i_1b_1}(\lambda^b)_{i_2b_2}, \\
\int dU U_{i_1a_1} U^\dagger{}_{j_1b_1} U_{i_2a_2} U^\dagger{}_{j_2b_2} U_{i_3a_3} U^\dagger{}_{j_3b_3} &= \frac{1}{3^3}\delta_{j_1a_1}\delta_{i_1b_1}\delta_{j_2a_2}\delta_{i_2b_2}\delta_{j_3a_3}\delta_{i_3a_3} \\
&\quad + \frac{1}{4\cdot 3\cdot 8}\left[(\lambda^a)_{j_1a_1}(\lambda^a)_{j_2a_2}(\lambda^b)_{i_1b_1}(\lambda^b)_{i_2b_2}\delta_{j_3a_3}\delta_{i_3a_3} + (3\leftrightarrow 1) + (3\leftrightarrow 2)\right] \\
&\quad + \frac{3}{8\cdot 5\cdot 8}d_{ijk}d_{abc}(\lambda^i)_{j_1a_1}(\lambda^j)_{j_2a_2}(\lambda^k)_{j_3a_3}(\lambda^a)_{i_1b_1}(\lambda^b)_{i_2b_2}(\lambda^c)_{i_3b_3} \\
&\quad + \frac{1}{8\cdot 3\cdot 8}f_{ijk}f_{abc}(\lambda^i)_{j_1a_1}(\lambda^j)_{j_2a_2}(\lambda^k)_{j_3a_3}(\lambda^a)_{i_1b_1}(\lambda^b)_{i_2b_2}(\lambda^c)_{i_3b_3}.
\end{aligned} \tag{E.2}$$

E.3 Fourier transform and low momenta approximations

Fourier transform is defined as follows:

$$f(p) = \int d^4x \, e^{ipx} f(x), \qquad f(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} f(p).$$

In terms of the modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$, the Fourier transform of the functions $\phi(x)$, $\Gamma(x)$, $\Lambda(x)$, $\chi(x)$ introduced in eq. (C.13) read:

$$\phi(p) = 2\pi\rho\left[I_0\left(\frac{u\rho}{2}\right)K_0\left(\frac{u\rho}{2}\right) - I_1\left(\frac{u\rho}{2}\right)K_1\left(\frac{u\rho}{2}\right)\right],$$

$$
\begin{aligned}
\Gamma_{\alpha\sigma}(p) &= \frac{32\pi\rho}{15}\,\theta_{\alpha\sigma}\left(\frac{d}{d\rho^2}\right)^2 I_1\left(\frac{u\rho}{2}\right)K_1\left(\frac{u\rho}{2}\right),\\
\Lambda_{\alpha\sigma}(p) &= -\frac{16\pi\rho^3}{15u}\,\theta_{\alpha\sigma}\frac{d}{du}\left(\frac{d}{d\rho^2}\right)^3 I_0\left(\frac{u\rho}{2}\right)K_0\left(\frac{u\rho}{2}\right),\\
\chi_{\alpha\sigma}(p) &= \frac{8i\pi}{45u\rho}\Bigg\{(\theta-\tilde{\theta})_{\alpha\sigma}\left[4\left[\left(\frac{d}{d\rho^2}\right)^2+\frac{3}{u}\frac{d}{du}\right]^2-14\rho^2\left[\left(\frac{d}{d\rho^2}\right)^2+\frac{3}{u}\frac{d}{du}\right]\right]\\
&\qquad\qquad -6(\theta+\tilde{\theta})_{\alpha\sigma}\Bigg\}\times\\
&\qquad \frac{d}{du}\left(\frac{d}{d\rho^2}\right)^3 I_0\left(\frac{u\rho}{2}\right)K_0\left(\frac{u\rho}{2}\right).
\end{aligned}
\tag{E.3}
$$

The variable u stands for $\sqrt{p^2}$. Let $'$ denote derivative with respect to u. We have the following low momenta — $u\rho \ll 1$ — expansions:

$$
\begin{aligned}
\phi'(u) &\sim -\frac{2\pi\rho}{u}+O(u),\\
\Gamma'''_{\alpha\sigma}(u) &\sim O(u),\ \Gamma''(u)\sim O(u^0),\ \Gamma'(u)\sim O(u),\\
\Lambda'_{\alpha\sigma}(u) &\sim O(u),\\
\chi'_{\alpha\sigma}(u) &\sim -\frac{2i\pi}{3\rho u}\,(\theta-\tilde{\theta})_{\alpha\sigma}+O(u).
\end{aligned}
\tag{E.4}
$$

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